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Stochastic Growth in a Small World and Applications to Scalable Parallel Discrete-Event Simulations

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Abstract

We consider a simple stochastic growth model on a small-world network. The same process on a regular lattice exhibits kinetic roughening governed by the Kardar-Parisi-Zhang equation. In contrast, when the interaction topology is extended to include a ***finite*** number of random links per site, the surface becomes macroscopically smooth. The correlation length of the surface fluctuations becomes finite and the surface grows in a mean-field fashion. Our finding provides a possible way to establish control ***without*** global intervention in non-frustrated agent-based systems. A recent application is the construction of a fully scalable algorithm for parallel discrete-event simulation.

Phase transitions in Small-World (SW) Networks

- Watts&Strogatz (1998): "… enhanced signal-propagation speed, ***computational power***, and ***synchronizability***".
- ***Finite number*** of random links per site (average degree is not extensive)
- ***Phase transition*** or phase ordering is possible even when random links are added to an originally ***one-dimensional*** substrate:
- Barrat&Weight (2000), Gitterman (2000), Kim et al. (2001), Herrero (2002), Jeong et al. (2003), Novotny and Wheeler (2004): Ising model on SW network
- Hong et al. (2002): XY-model and Kuramoto oscillators on SW ntwk.
- Hastings (2003): general criterion for ***mean-field***-like phase transitions for interacting systems on SW networks.

Synchronization in Parallel Discrete-Event Simulations

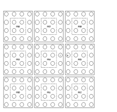
Parallelization for asynchronous dynamics

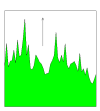
Paradoxical task:
- (algorithmically) parallelize (physically) non-parallel dynamics

Difficulties:
- Discrete events (updates) are not synchronized by a global clock
- Traditional algorithms appear inherently serial (e.g., Glauber attempt one site/spin update at a time)

However, these algorithms are not inherently serial (Lubachevsky '87)

Two Approaches for Synchronization



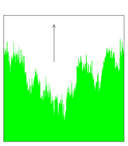
τ_i  d=1

(site index) i

❖**Optimistic (or speculative)**
- PEs assume no causality violations
- *Rollbacks* to previous states once causality violation is found (extensive state saving or reverse simulation)
- Rollbacks can cascade ("avalanches")

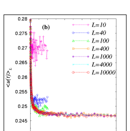
❖**Conservative**
- PE "idles" if causality is not guaranteed
- Utilization, $\langle u \rangle$: fraction of non-idling PEs

Basic Conservative Approach



- one-site-per PE, $N_{PE}=L^d$
- t=0,1,2,… parallel steps
- $\tau_i(t)$ local simulated time
- local time increments are iid exponential random variables
- advance only if $\tau_i \le \min\{\tau_{nn}\}$

(nn: nearest neighbors)



❖**Scalability modeling**
- ***utilization*** (efficiency) $\langle u(t)\rangle$ (fraction of non-idling PEs) density of local minima
- ***width*** (spread) of time surface:

$$w^2(t)=\frac{1}{N_{PE}}\sum_{i=1}^{N_{PE}}[\tau_i(t)-\bar{\tau}(t)]^2$$

Simulating the Parallel Simulations

❖Universality/roughness (d=1)

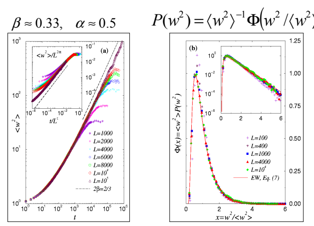
$$\langle w^2(t)\rangle_L \sim \begin{cases} t^{2\beta}, & \text{if} \quad t \ll t_\times \\ L^{2\alpha}, & \text{if} \quad t \gg t_\times \end{cases}, \quad t_\times \sim L^z, \quad z=\alpha/\beta$$

Foltin et.al., '94

$\beta \approx 0.33, \quad \alpha \approx 0.5$ $\qquad P(w^2)=\langle w^2\rangle^{-1}\Phi(w^2/\langle w^2\rangle)$

exact KPZ:
β=1/3
α=1/2

G.K. et al., *PRL* **84**, 1351 (2000)



Utilization (Efficiency)

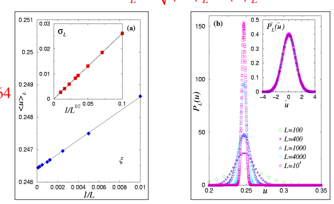
Finite-size effects for the density of local minima/average growth rate (steady state):

(d=1)

$$\langle u\rangle_L \cong \langle u\rangle_\infty + \frac{const.}{L}$$

$$\sigma_L=\sqrt{\langle u^2\rangle_L-\langle u\rangle_L^2} \sim 1/L^{1/2}$$

$\langle u\rangle_\infty \approx 0.2464$



Implications for Scalability

Simulation reaches steady state for $t \gg L^z$ (arbitrary d)

❖Simulation phase: scalable $\qquad \langle u\rangle_L \cong \langle u\rangle_\infty + \frac{const.}{L^{2(1-\alpha)}}$

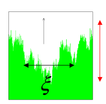
$\langle u\rangle_\infty$ asymptotic average growth rate (simulation speed or utilization) is non-zero

Krug and Meakin, '90

❖Measurement (data management) phase: not scalable

measurement at τ_{meas}:
(e.g., simple averages)

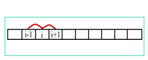
rough (self-affine) synchronization landscape



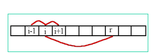
$w \sim L^\alpha$

$\xi \sim L$

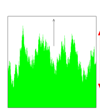
Synchronization/Time-Horizon Control Via Small-World Communication Network Design



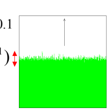
small-world-like connections: (used with probability p>0)



$p=0.0$ $\qquad$ $p=0.1$



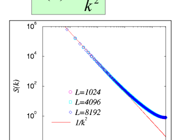
$w \sim L^\alpha$ $\qquad$ $w=const.+o(L^{-1})$

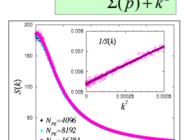


1d (ring) $\qquad$ **steady-state structure factor: (Fourier transform)** $\qquad$ SW network

$$S(k) \propto \langle \tau_k \tau_{-k}\rangle = \langle |\tau_k|^2\rangle$$

$$S(k) \sim \frac{1}{k^2} \qquad S(k) \sim \frac{1}{\Sigma(p)+k^2}$$





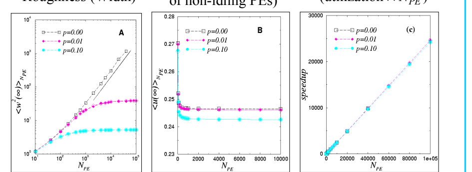
Utilization Trade-off/Scalable Data Management

$$\partial_t \tau = -\Sigma(p)\tau + \frac{\partial^2 \tau}{\partial x^2} + \ldots + noise$$

G.K. et al., *Science* **299**, 677 (2003)

effective relaxation to the mean facilitated by the SW links

Roughness (Width) $\qquad$ Utilization (fraction of non-idling PEs) $\qquad$ Speedup (utilization × N_{PE})



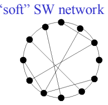
Edwards-Wilkinson Model on a Small-World Network

$$\partial_t h_i = -(2h_i - h_{i+1} - h_{i-1}) - \sum_{j=1}^{N} J_{ij}(h_i - h_j) + \eta_i(t)$$

$$\partial_t h_i = -\sum_{j=1}^{N} \Gamma_{ij} h_j + \eta_i(t) \qquad \Gamma_{ij}^o = 2\delta_{ij} - \delta_{i\,j+1} - \delta_{i\,j-1}$$

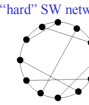
$$\Gamma = \Gamma^o + V \qquad V_{ij} = \delta_{ij}\sum_l J_{il} - J_{ij}$$

"soft" SW network (Erdős-Rényi network on top of the ring):



$$J_{ij} = \begin{cases} 1 & \text{with probability} \quad p/N \\ 0 & \text{with probability} \quad 1-p/N \end{cases}$$

"hard" SW network:



$J_{ij}=0$ or p: N/2 random links are selected, such that each site has exactly one random link of strength p (in addition to n.n.)

Width from exact numerical diagonalization:

$\{\lambda_l\}_{l=0}^{N-1}$ eigenvalues of Γ_{ij} $\qquad (\lambda_0 = 0)$

$$\langle w^2\rangle_N = \left\langle \frac{1}{N}\sum_{i=1}^{N}(h_i-\bar{h})^2 \right\rangle = \frac{1}{N}\sum_{l=1}^{N-1}\frac{1}{\lambda_l}$$

for a single realization of the random network

$[\langle w^2\rangle_N]$ averaged over network realizations ("disorder-averaged" width)

Impurity-averaged perturbation theory

Kozma et al. *PRL* **92**, 108701 (2004).

$$[G]^{-1} = \Gamma^o + \Sigma \qquad G=\Gamma^{-1} \qquad G^o = \Gamma^{o^{-1}}$$

self-energy (effective interaction due to random links)

$$[\langle w^2\rangle_N] = [(\Gamma^{-1})_{ii}]$$

"soft" network: $\Sigma \sim p^2 + \ldots$ $\qquad$ [see also Monasson, EPJB **12**, 555 (1999) in the context of diffusion on SWN]

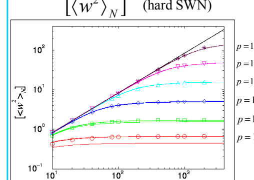
"hard" network: $\Sigma \sim p - \frac{1}{2}p^{3/2} + \ldots$

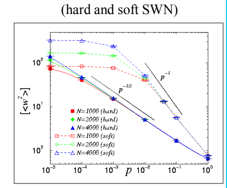
$$\xi \overset{N\to\infty}{\sim} \frac{1}{\sqrt{\Sigma}} \qquad [\langle w^2\rangle_N] \overset{N\to\infty}{\sim} \frac{1}{2\sqrt{\Sigma}}$$

finite correlation length $\qquad$ for any p>0 $\qquad$ finite width

Comparison of exact numerical diagonalization of Γ_{ij} with the results of the impurity-averaged perturbation theory

$[\langle w^2\rangle_N]$ (hard SWN) $\qquad$ (hard and soft SWN)





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Summary

- Synchronizability of large-scale non-frustrated agent-based systems with **SW** network: application to construct fully scalable parallel simulations ***without*** global synchronizations
- Spectrum of the coupling matrix exhibits a gap/pseudo-gap, yielding a ***finite width*** for stochastic growth on a small-world network ***for all*** p>0