

Quantum Monte Carlo methods: recent developments and applications

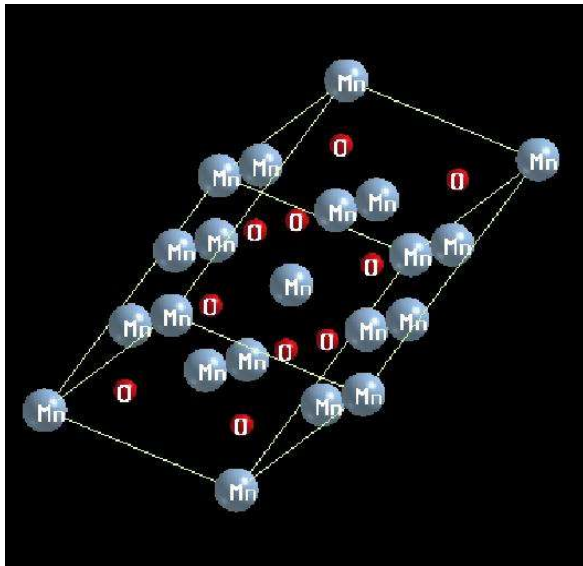
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Urbana, August 2006



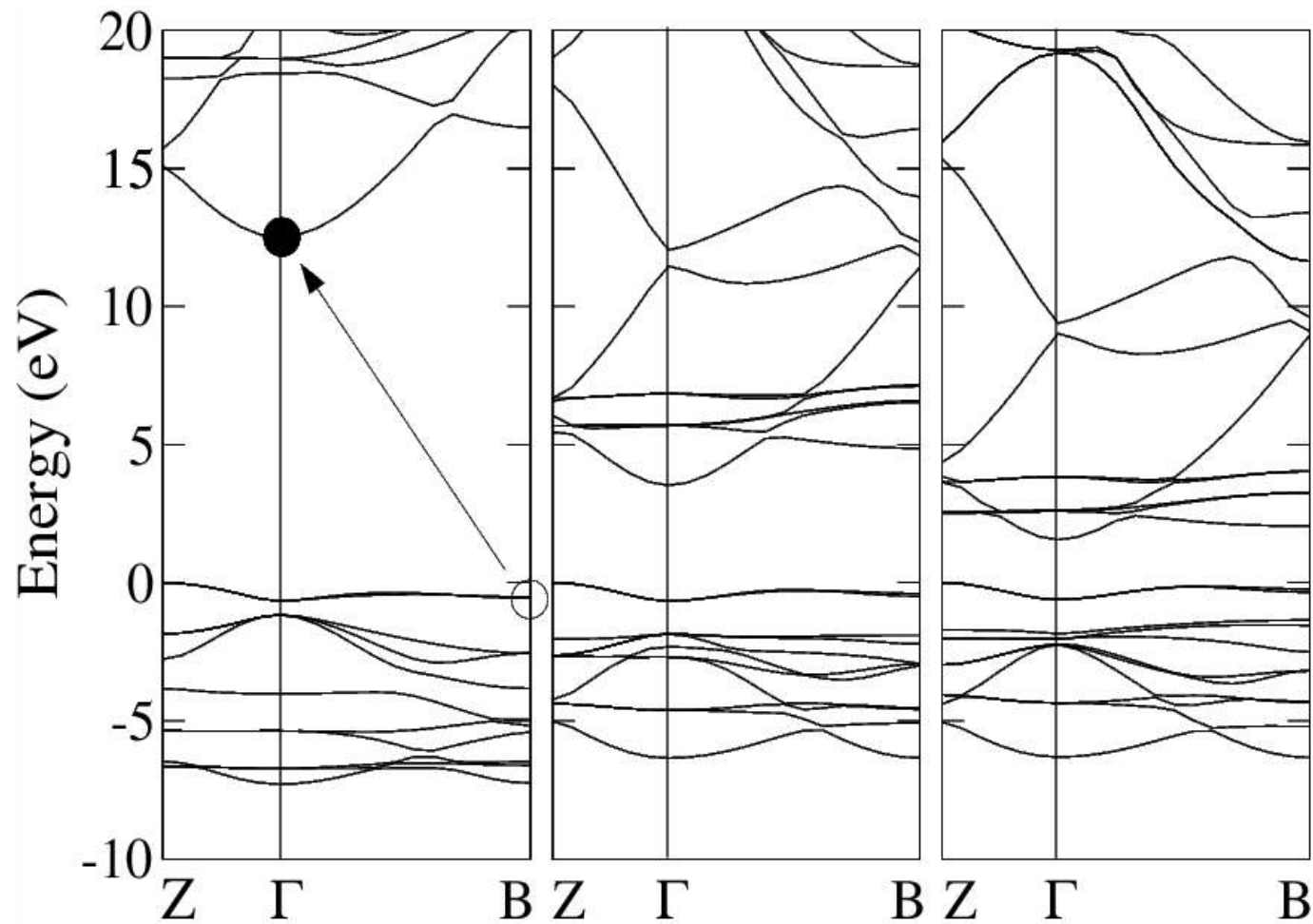
MnO solid calculations: paradigmatic TMO antiferromagnet



- large correlation effects, difficult to treat by traditional methods: competition of Coulomb, exchange, correlation and crystal-field effects
- supercells with 8, 16, 20 and 24 atoms: up to ~ 250 valence electrons
- gap estimations using promotion from valence -> to conduction band

Band structure of MnO by HF and DFT:

UHF B3LYP PW91



MnO cohesion and band gap

Cohesive energy [eV]

HF	B3LYP	DMC	Exp.
6.0	10.2	9.40(5)	9.5

Band gap: B \rightarrow Γ μ μ α excitation

HF	B3LYP	DMC	Exp.
14.2	4.0	4.8(3)	4.2

Small bias towards higher energy for the excited state

BaTiO₃ cohesion

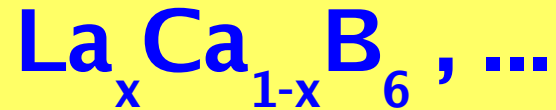
Cohesive energy [eV]

DFT/PW91	DMC/HF	DMC/DFT	Exp.
38.0	30.9(4)	31.2(4)	31.56

Note: cohesion comes very close to experiment from a straightforward application of QMC

Current effort: ferroelectric distortion

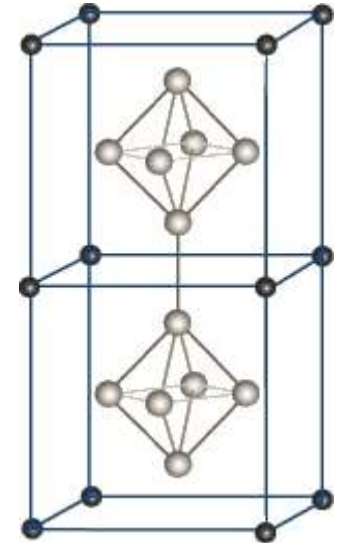
Electronic structure of hexaborides CaB_6 ,



3% La-doped CaB_6 is a weak magnet up to 600-900K (!)

No *d* or *f* electrons: - genuine itinerant magnetism ?

Undoped CaB_6 : insulator ? excitonic insulator ? metal ?

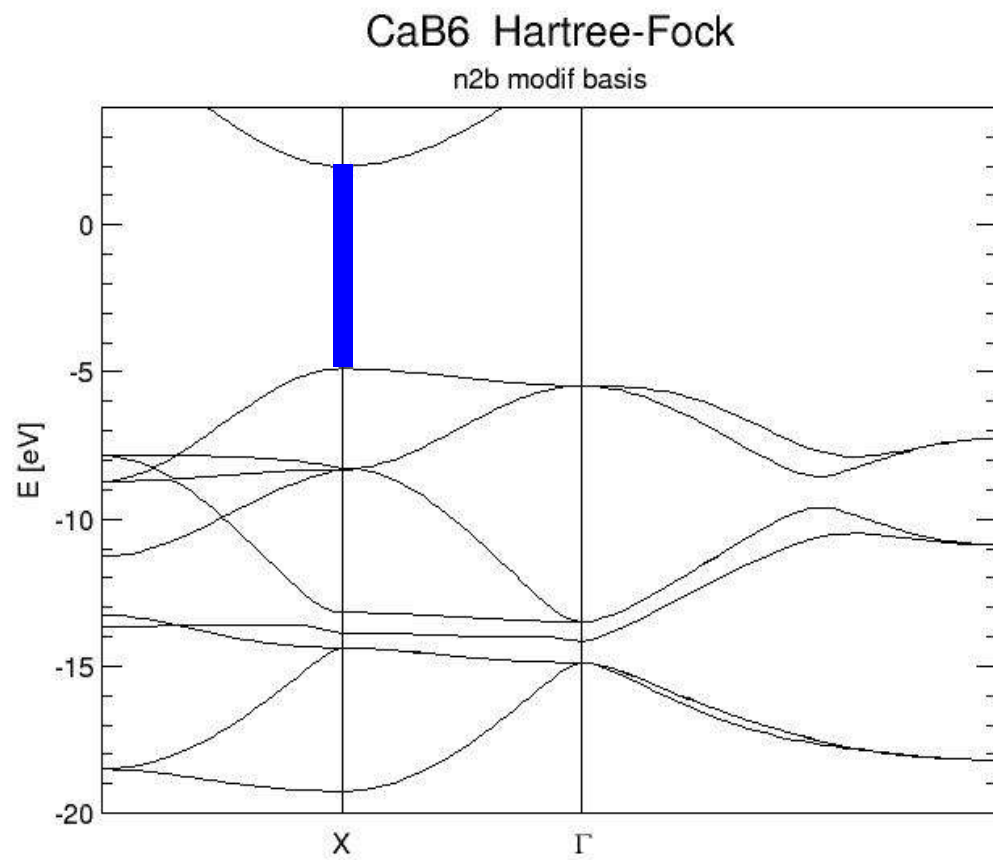


Experiments contradictory:
both metallic and insulating
behavior observed

Calculations inconclusive:
semimetal or insulator

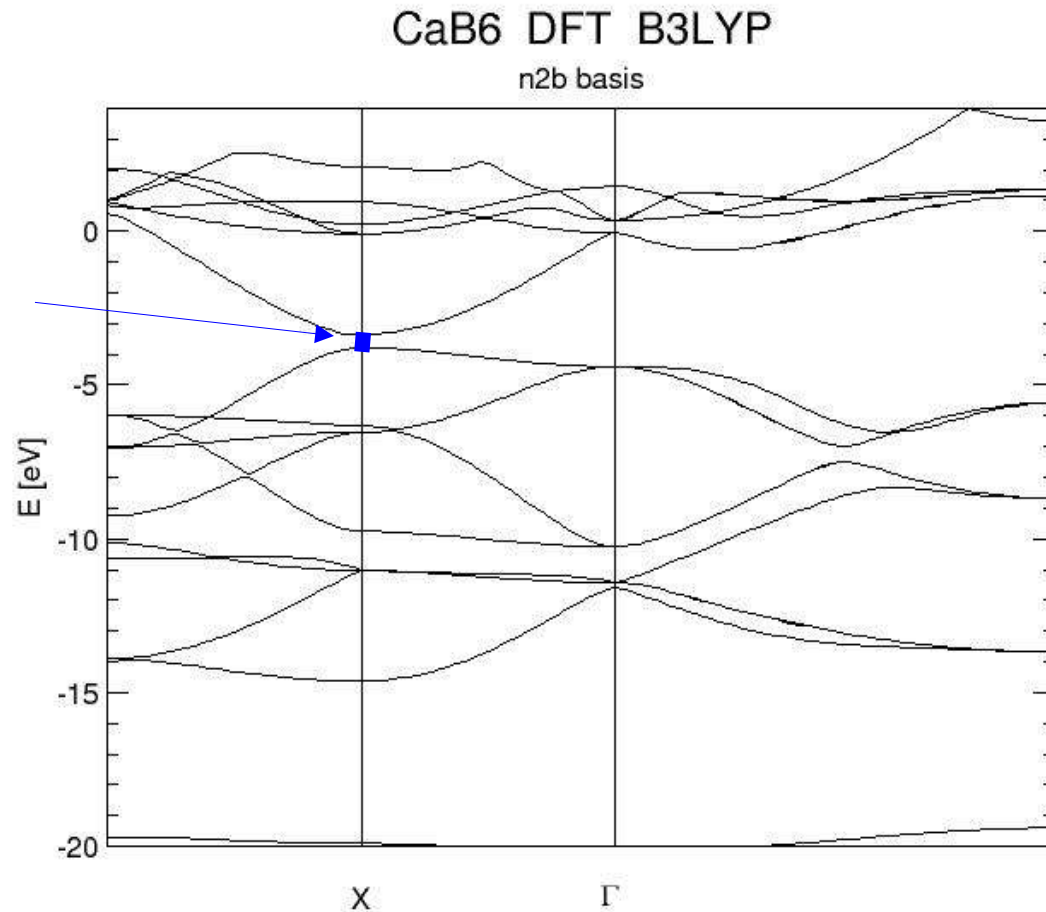
Can we calculate the gap before the experiment ?

CaB₆ band structure in Hartree-Fock



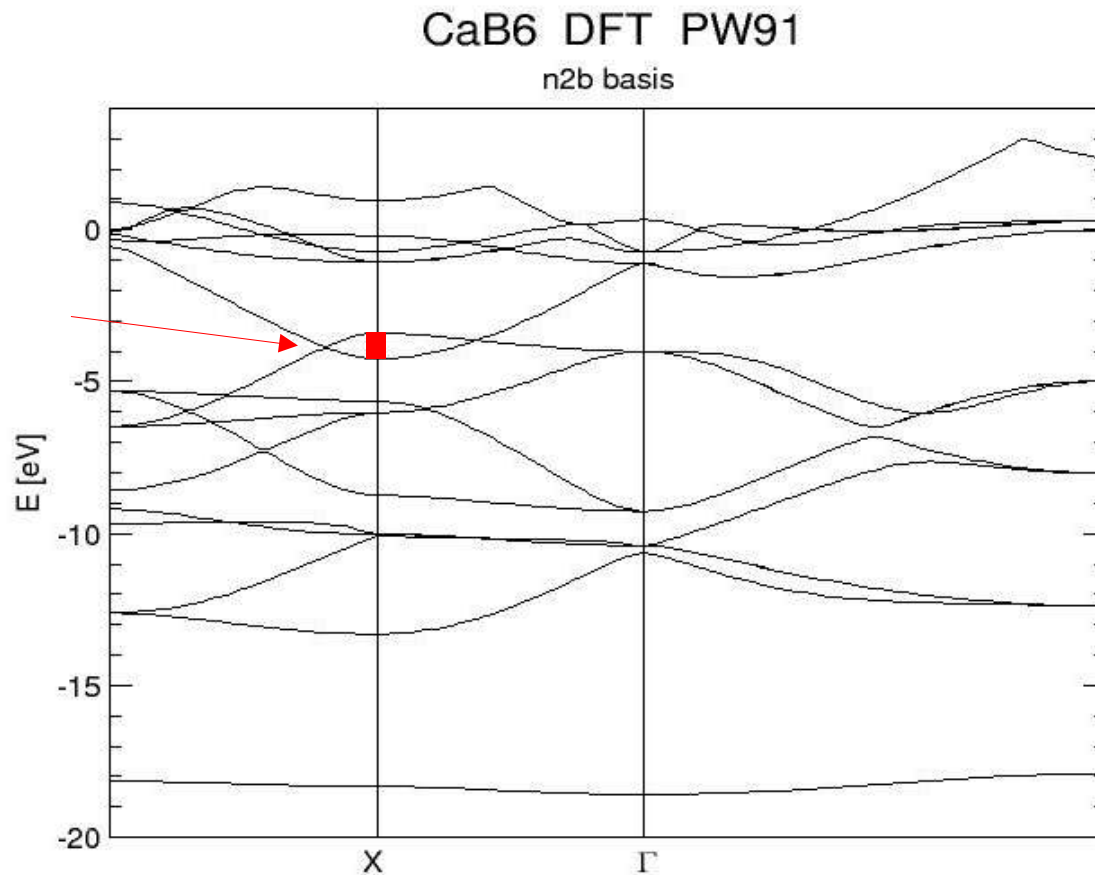
**Large gap
of the order of 7 eV**

CaB₆ band structure in DFT - B3LYP



Gap is now only
about 0.5 eV

CaB₆ band structure in DFT - PW91

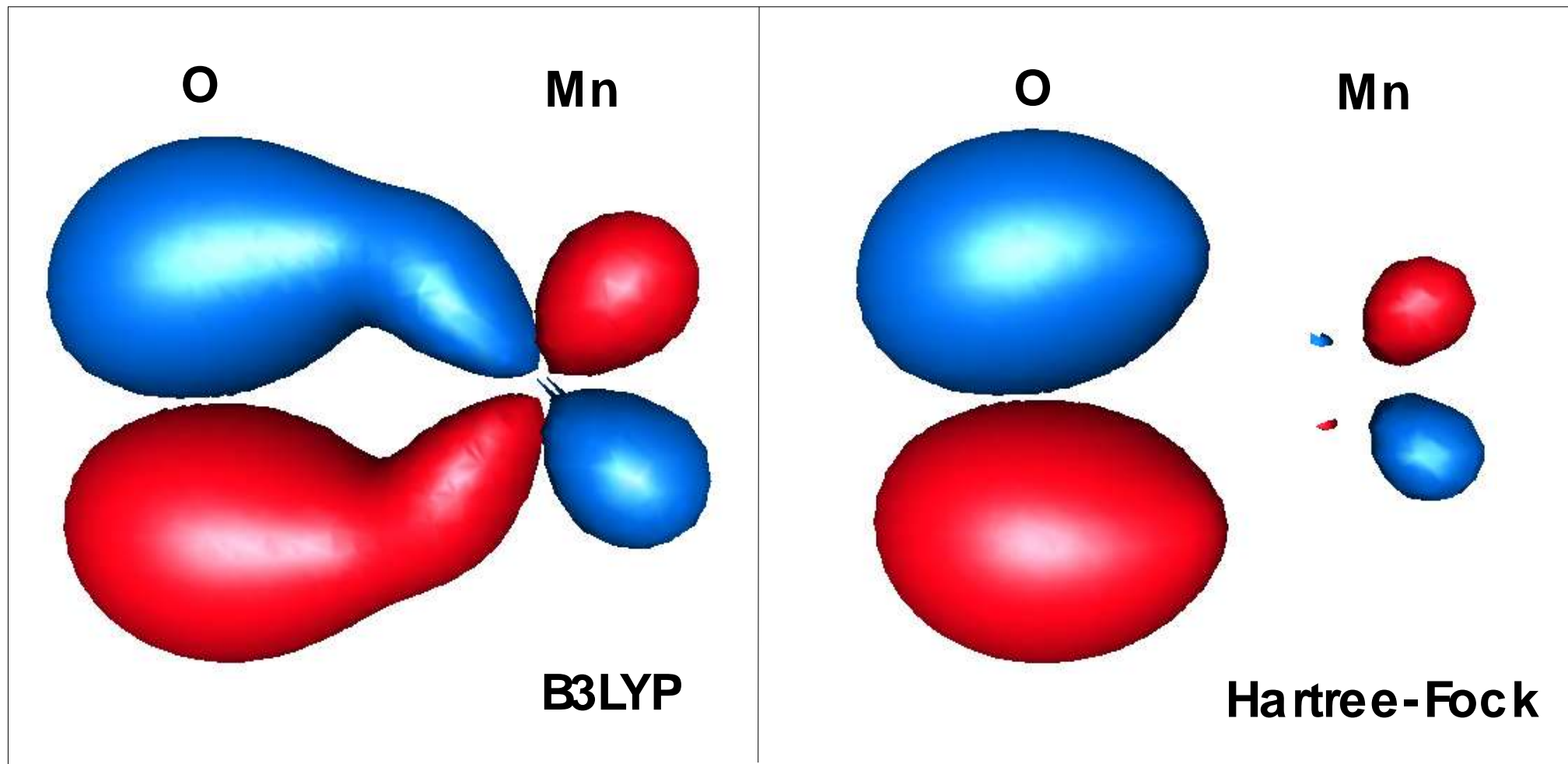


**1 eV overlap at the
X point:
d-states on Ca**

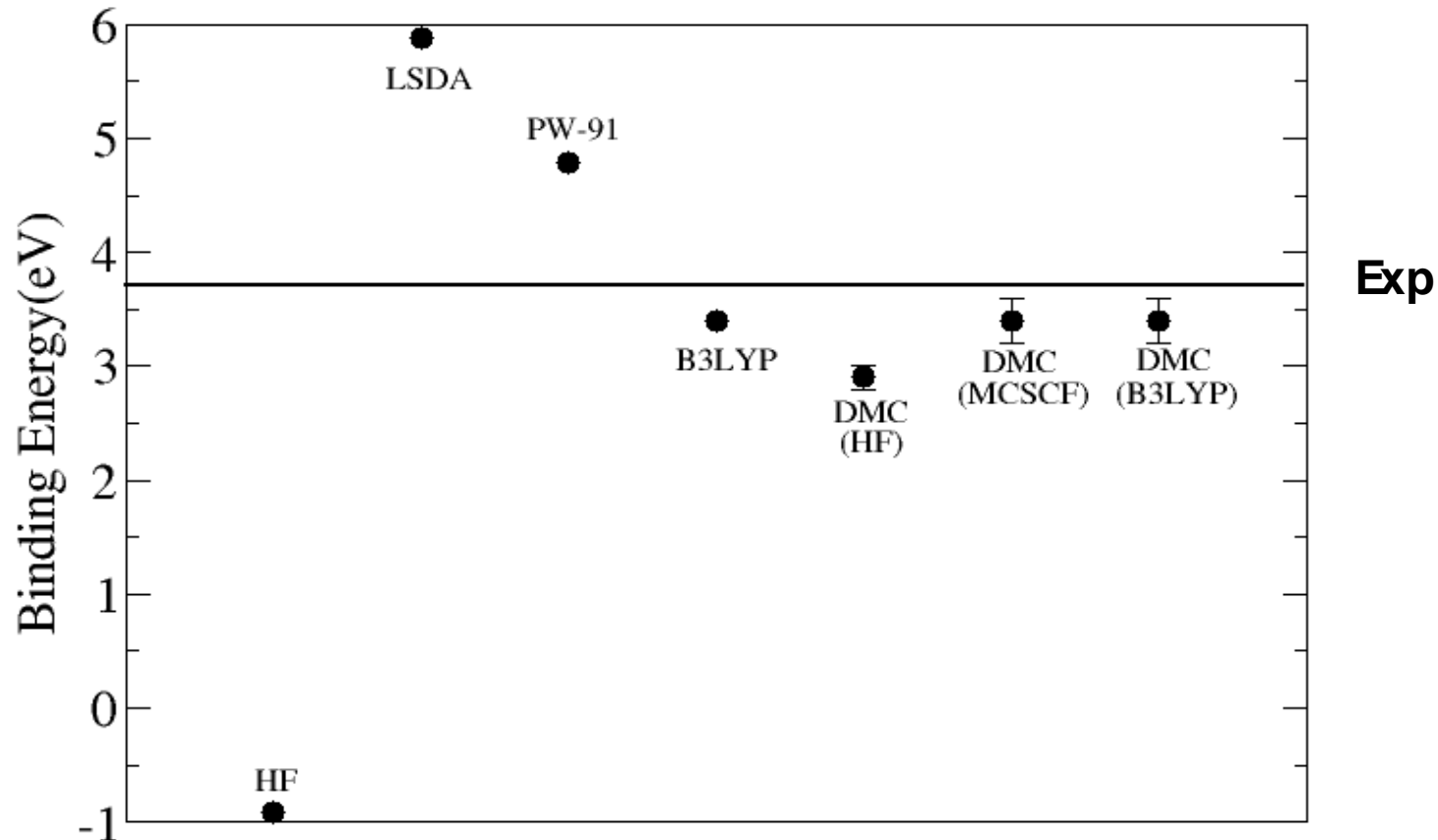
**Fixed-node
DMC gap:**

1.2(2) eV

Oxygen 2p-3d hybridized orbitals: HF vs B3LYP, ie, charge transfer vs covalent bonding



MnO binding by various methods

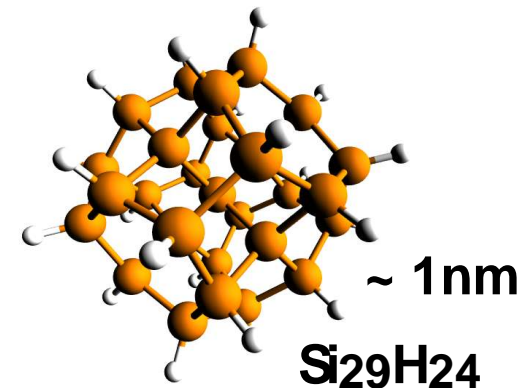


FNDMC with B3LYP orbitals within 0.2 eV of experiment

Silicon nanocrystals: ultrabright luminescence quantum dots

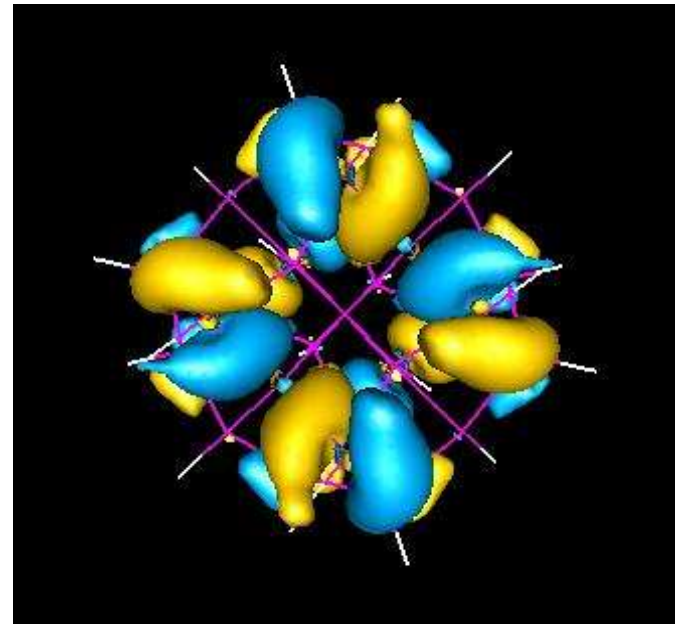
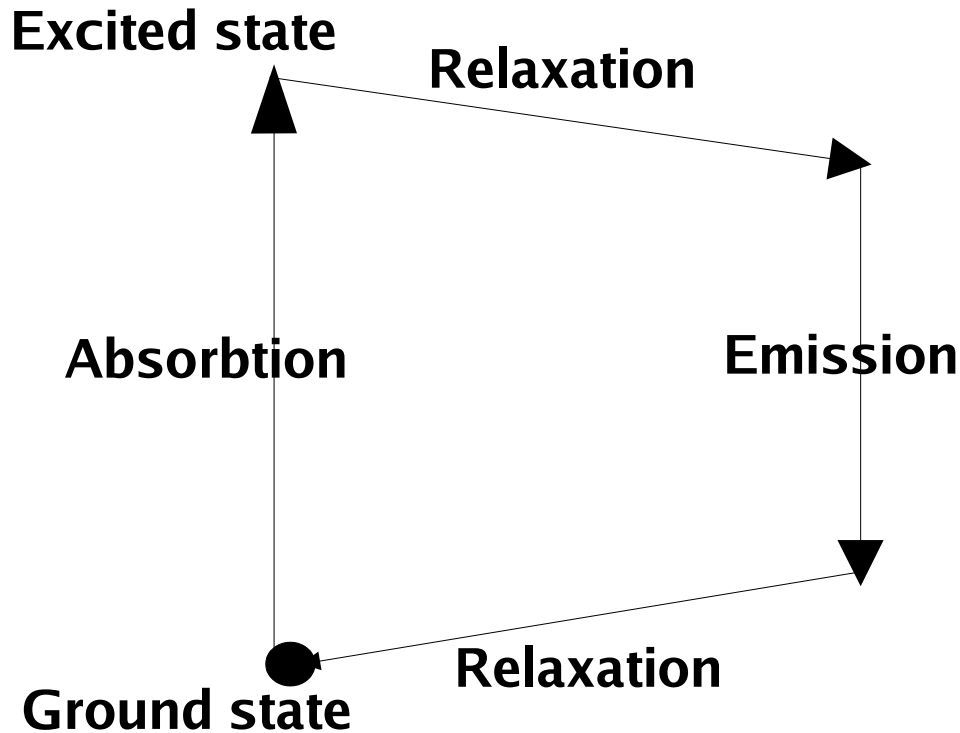
Collaboration with experimental group at U. Illinois (M.Nayfeh) and LLNL group (Grossman, Williamson, Puzder)

- unlike bulk silicon, Si nanocrystals have interesting optical properties: strong luminescence in the visible range
- can be prepared in large quantities and in reasonably homogeneous batches of sizes from 1 nm ("blue") to about 2.7 nm ("red"), "magic sizes"
- nanocrystals are stable with hydrogen passivated or reconstructed surfaces
- lots of interest: sensors, optical devices, biomarkers,...



Optical excitons

Stokes shift = $E_{\text{absorption}} - E_{\text{emission}}$



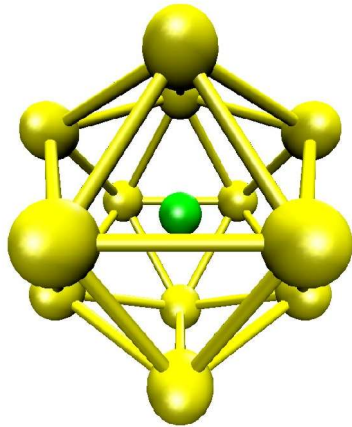
DMC	0.4(1) eV
Exper.	0.5(1) eV

(Collaboration with A. Puzder, A. Williamson, J. Grossman, LLNL)

Magnetic "nanodots": caged transition elements $TM@Si_{12}$ $TM=Sc, Ti, \dots 3d, 4d, 5d$

APS March Meeting in '94:

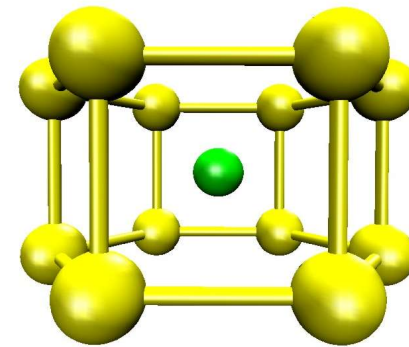
L. M.: Electronic structure of $Mn@Si_{12}$



- attempt to predict caged d-spin
- no success, hybridized, unstable

Experiment in Japan in '01!

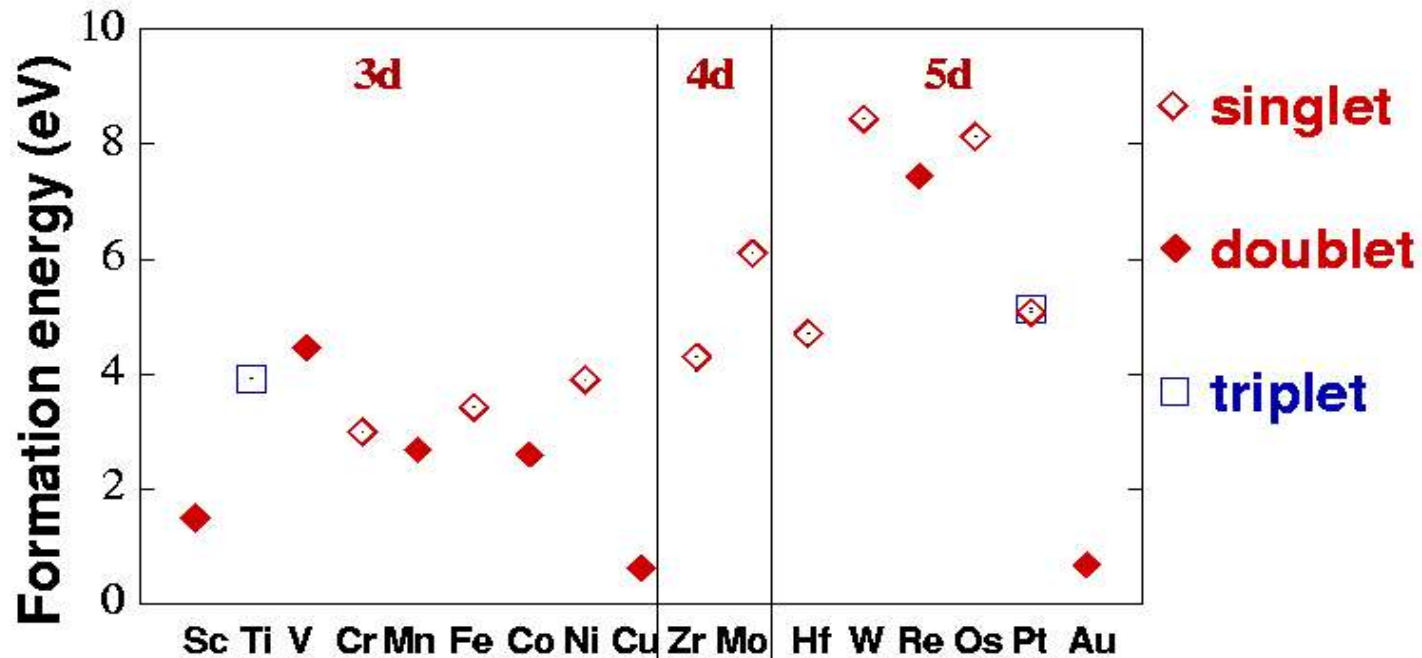
$W@Si_{12}$



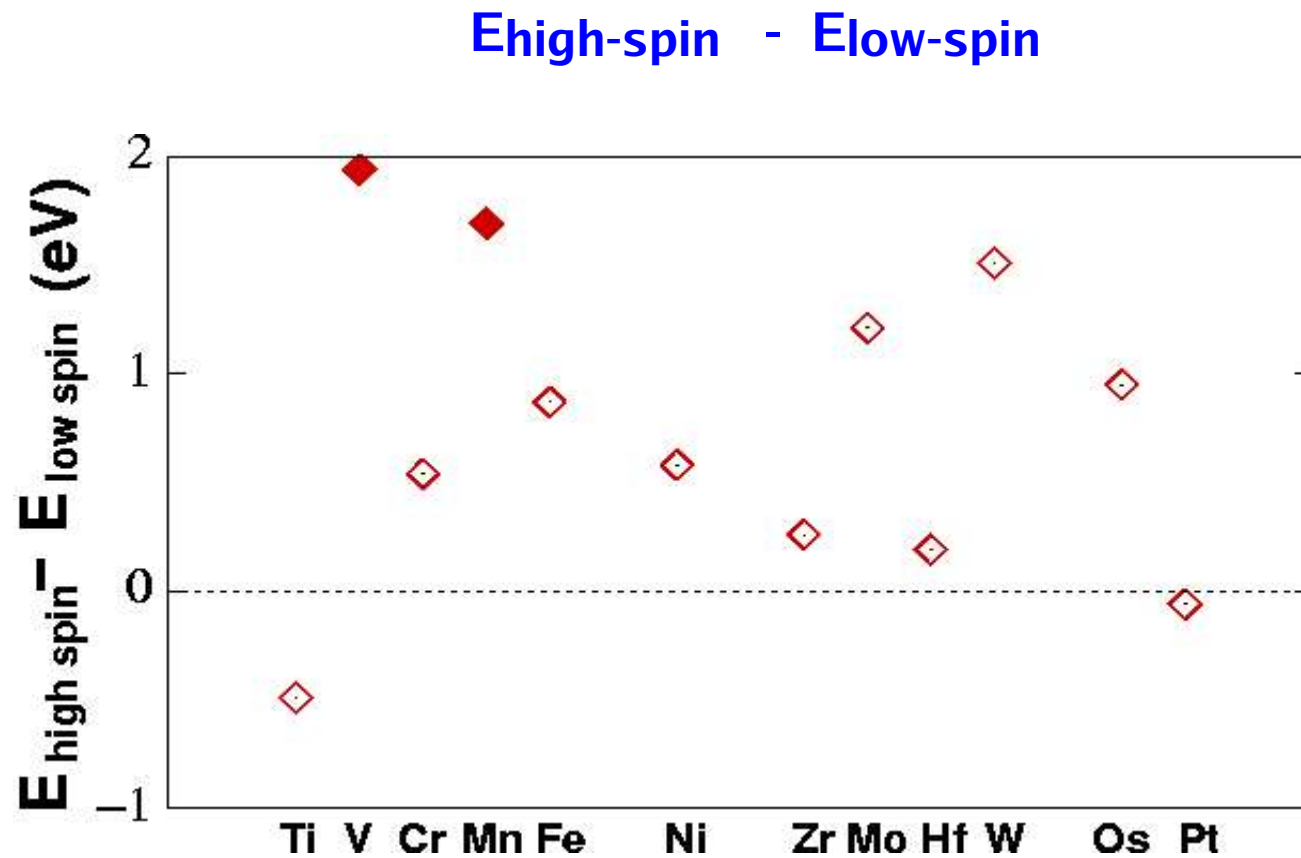
Other transition metal atoms ?

Key issue: stability for various TMs

$$\text{Formation energy} = E(\text{Si}_{12}, \text{lowest isomer}) + E(\text{TM}) - E(\text{TM}@ \text{Si}_{12})$$

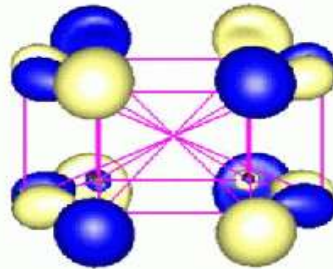


Difference high-spin vs low-spin state: prediction of Si_{12}Ti as a high-spin system

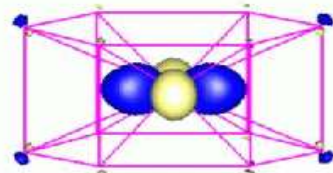
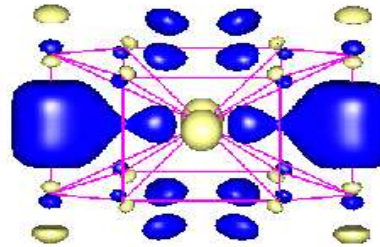


Different p-d hybridization for 3d-4d-5d: tunable difference between multiplets

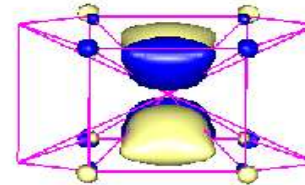
TiSi₁₂
triplet
ground
state



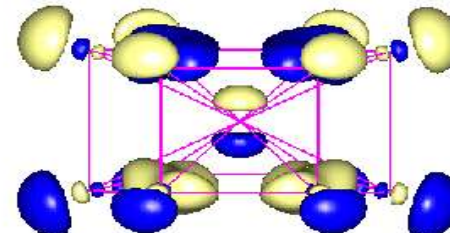
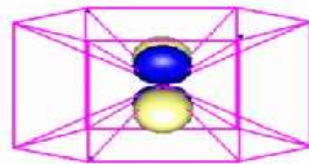
LUMO



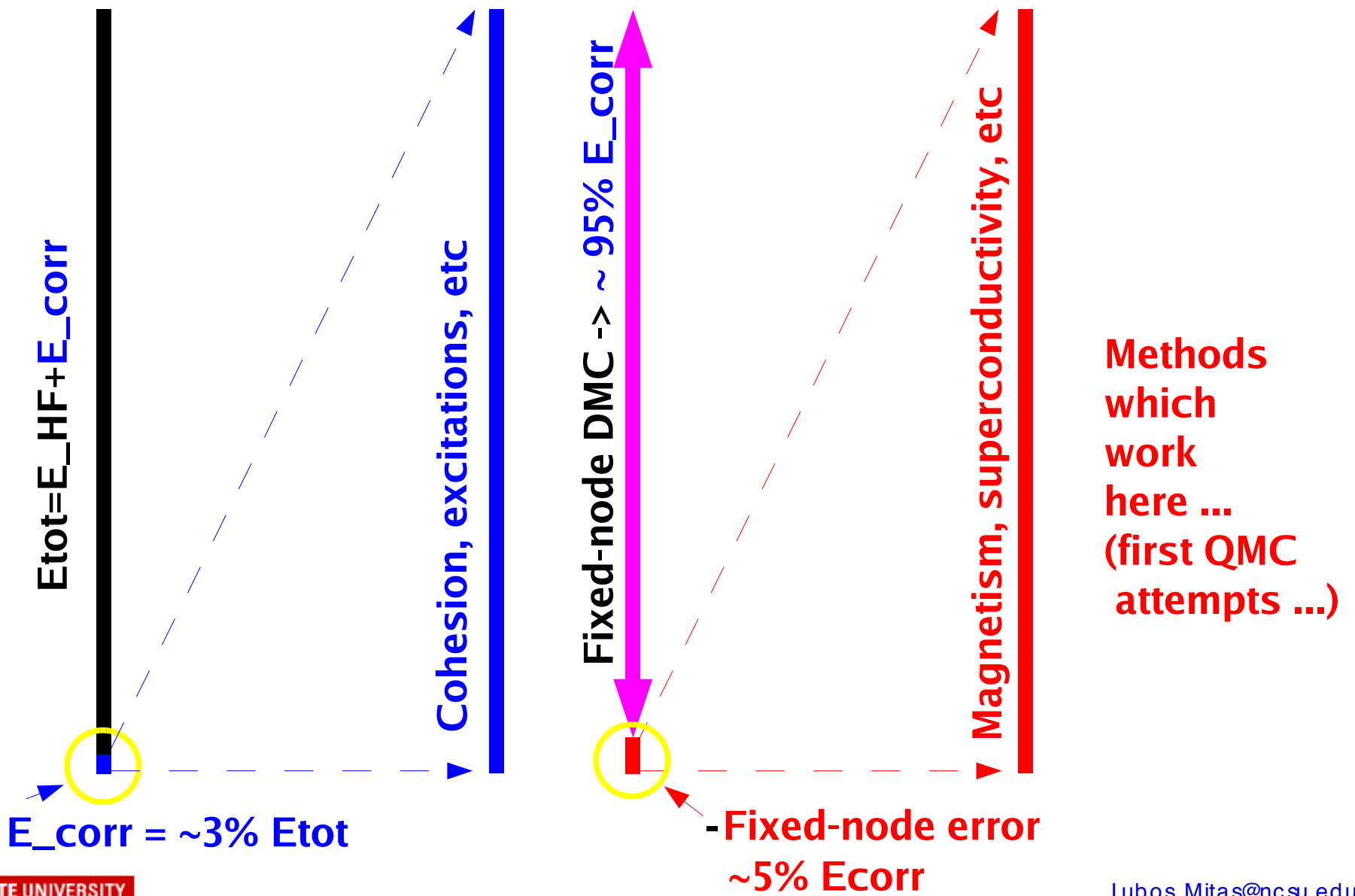
HOMO



HfSi₁₂
singlet
ground
state



Back to fundamentals: beyond the fixed-node DMC



“ Capture the nodes/physics” strategy calls for better wavefunctions: BCS wavefunction

- antisymmetized product of singlet pair orbitals $\phi(i, j)$

$$\psi_{BCS} = A[\phi(i, j)] = \det[\phi(i, j)]$$

for a singlet state (Sorella and coworkers, '03)

- **spin-polarized case:** $N^\downarrow = n$ while $N^\uparrow = n + o$

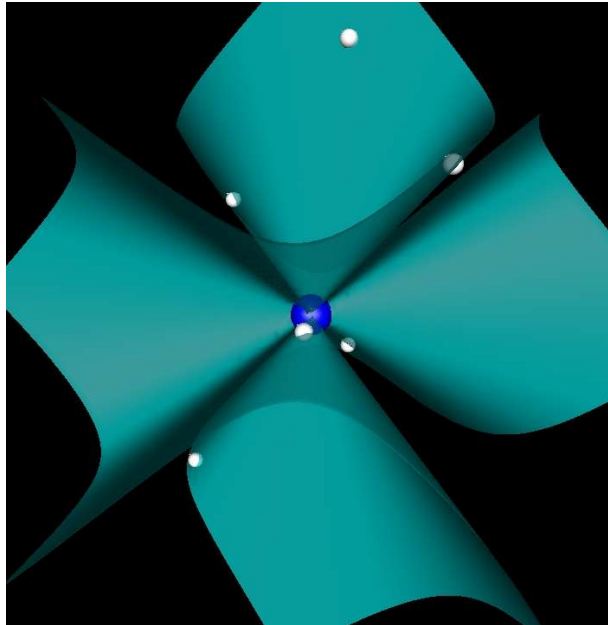
$$\psi_{BCS} = A[\phi(1, n) \dots \phi(n, 2n) \times h_1(2n+1) \dots h_o(2n+o)]$$

where $h_k(i)$ are one-particle orbitals (usually HF)

- for fully spin-polarized state recovers Hartree-Fock

$$\psi_{BCS} = A[h_i(j)] = \det[h_i(j)] = \psi_{HF}$$

Structure of fermion nodes and pfaffian pairing wavefunctions



Solving many-particle Schrodinger equation by fixed-node diffusion Monte Carlo (FNDMC)

$$f(\mathbf{R}, t+\tau) = \int G^*(\mathbf{R}, \mathbf{R}', \tau) f(\mathbf{R}', t) d\mathbf{R}'$$

$$f(\mathbf{R}, t) = \psi_T(\mathbf{R}) \phi(\mathbf{R}, t), \quad \psi_T = \psi_{HF} e^{U_{corr}} = \det\{\phi_\alpha\} \det\{\phi_\beta\} e^{U_{corr}}$$

$$G^*(\mathbf{R}, \mathbf{R}', \tau) = \frac{\langle \mathbf{R} | \exp(-\tau H) | \mathbf{R}' \rangle}{\psi_T(\mathbf{R}') \psi_T^{-1}(\mathbf{R})}$$

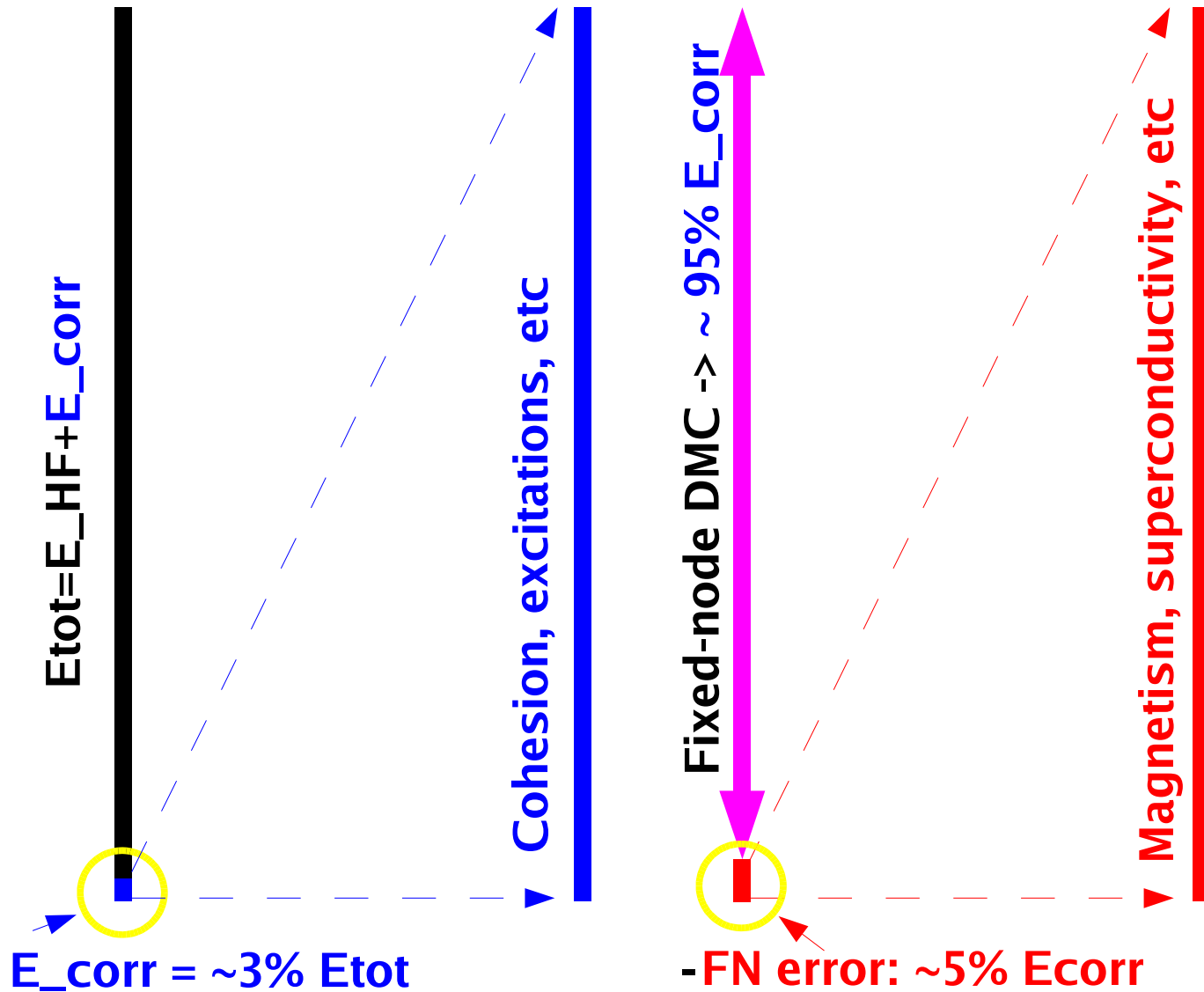
$$\lim_{\tau \rightarrow \infty} f(\mathbf{R}, \tau) \propto \psi_T(\mathbf{R}) \phi_{ground}(\mathbf{R})$$

Fermion node: defined as $\phi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = 0$

Fixed-node approximation: $f(\mathbf{R}, t) > 0$

Antisymmetry (nonlocal) replaced by a boundary (local)

Experience with the fixed-node DMC: applicable to tens/hundreds of valence electrons – but ...



Methods which work here ???
(beyond the fixed-node ...)

Fermion node: manifold of configurations for which the wave function vanishes

The (only) approximation in quantum Monte Carlo

Fermion node: $\phi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = 0$ (dN-1)-dimen. hypersurface

Fixed-node approximation: $f(\mathbf{R}, t) > 0$ (boundary replaces antisymmetry)

The Schrodinger eq.

$$f(\mathbf{R}, t + \tau) = \int G^*(\mathbf{R}, \mathbf{R}', \tau) f(\mathbf{R}', t) d\mathbf{R}'$$

$$f(\mathbf{R}, t \rightarrow \infty) = \psi_{Trial}(\mathbf{R}) \phi_{Ground}(\mathbf{R})$$

Exact node -> exact energy in polynomial time

Exact nodes:

- in 1D, particle coincidence points
- in 3D known for a few 2e and 3e states

In general, high-dimensional problem influenced by many-body effects and interactions

Antisymmetry/fermion sign problem: various ideas how to deal with the nodes

“Sample-it-out”:

- nodal release (Ceperley '80s)
- walker pairing algorithms (Kalos '90s)
- transform into another space (Hubbard -Stratonovitch) ...

“Capture the physics (the nodes will follow)”:

- more elaborate wavefunctions
- backflow
- pair orbitals, pfaffians, ...

“Understand the nodes”:

- general properties
- new insights, more fundamental issue (?)



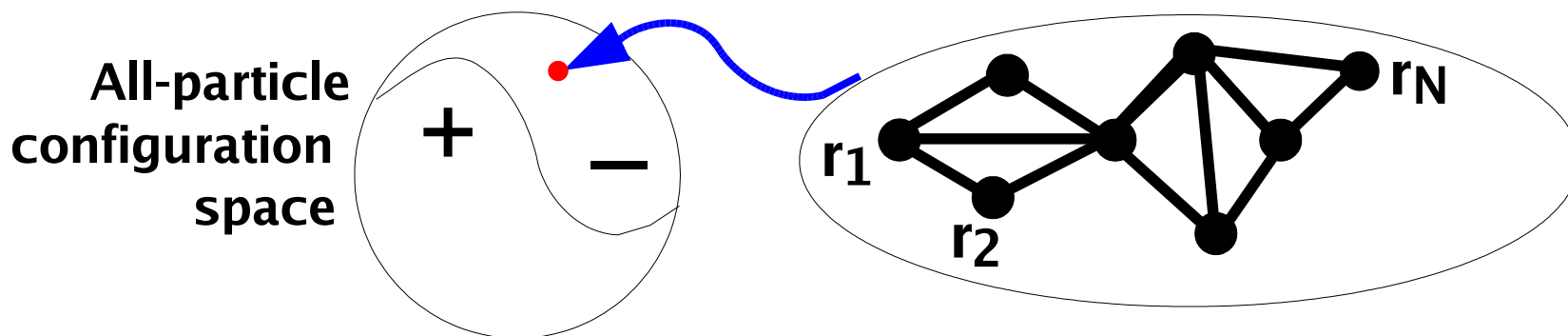
Key questions: - correct topology, ie, number of nodal cells
- correct shape

Topology of a few-electron exact nodes, numerical studies -> **Conjecture:** for $d > 1$ ground states have only two nodal cells, one “+” and one “-”

So far unproven even for noninteracting systems !

Tool to demonstrate some $\psi(R)$ has only two cells (Ceperley '92, numerical proofs up to 200 noninteracting fermions):

Find a point such that triple exchanges connect all the particles into a single cluster: then there are only two nodal cells



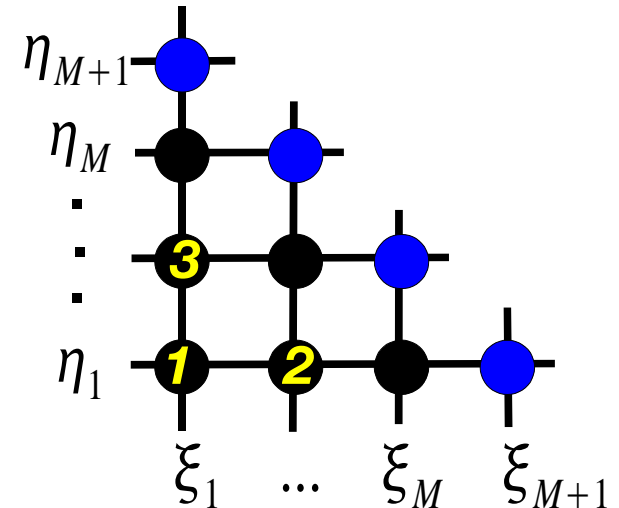
Explicit proof of two nodal cells for **spin-polarized** noninteracting system for any size (Idea illustrated for 2D harmonic fermions)

Place fermions in a Pascal-like triangle

M lines $\rightarrow N_M = (M+1)(M+2)/2$ particles

The wavefunction:

$$\psi_M(1, \dots, N_M) = C_{gauss} \det[1, x, y, x^2, xy, y^2, \dots]$$



Evaluated explicitly by recursion:
factorizing out “lines of particles”

$$\psi_M(1, \dots, N_M) = \psi_{M-1}(1, \dots, N_M / I_{\xi_1}) \prod_{i < j}^{i, j \in I_{\xi_1}} (x_j - x_i) \prod_{1 < k \leq M} (\xi_k - \xi_1)^{n_k}$$

By induction: if N_M particles are connected, then also N_{M+1} . QED.

The key points of the proof:

- a) Slater matrix elements are multivariate monomials
- b) configuration enables to factorize the determinant

The factorization enables to explicitly show that the particles are connected

Any model which transforms to homogeneous polynomials!

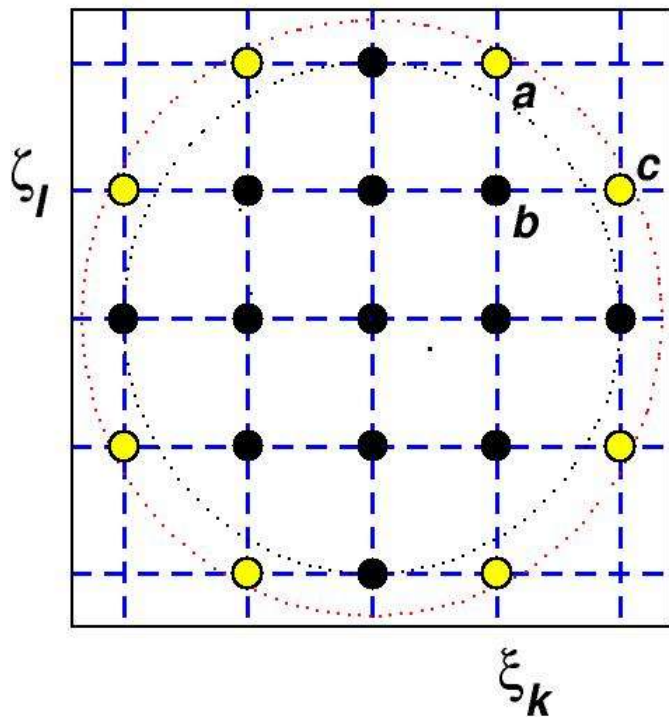
- fermions in a periodic box $\phi_{nm}(x, y) = e^{i(nx+my)} = z^n w^m$

- fermions on a spherical surface $Y_{lm}(\theta, \phi) = (\cos \theta)^n (\sin \theta e^{i\phi})^m$

- fermions in a box $\phi_{nm}(x, y) = \sin(x) \sin(y) U_{n-1}(p) U_{m-1}(q)$

homeomorphic variable map: $p = \cos(x), q = \cos(y) \rightarrow p^m q^n$

2D periodic fermions: similar factorization 3D or higher: the same idea



- particles on the line $x = \xi_k$

$$\psi_{1D}(\dots, i_l^{(k)}, \dots, i_m^{(k)}, \dots) = \prod_{l < m} \sin(\zeta_{lm}/2)$$

- complete wavefunction

$$\psi(1, \dots, N) = \prod_{k=0}^M \left[\psi_{1D}(I_k) \prod_{j < k} \sin^{n_j}(\xi_{jk}/2) \right]$$

- induction step: particles a,b,c connected

Factorization works for any $d > 1$ (!!!): lines, planes, hyperplanes

Two nodal cells theorem: generic (and fundamental) property of fermionic ground states of many models

Two nodal cells theorem. Consider a spin-polarized system with a closed-shell ground state given by a Slater determinant times an arbitrary prefactor (which does not affect the nodes)

$$\psi_{exact} = C(1, \dots, N) \det \{ \phi_i(j) \}$$

Let the Slater matrix elements be monomials of positions or their homeomorphic maps. $x_i^n y_i^m z_i^l \dots$

Then the wavefunction has only two nodal cells.

With some effort can be generalized to some open shells.

What if matrix elements are **not** monomials ?

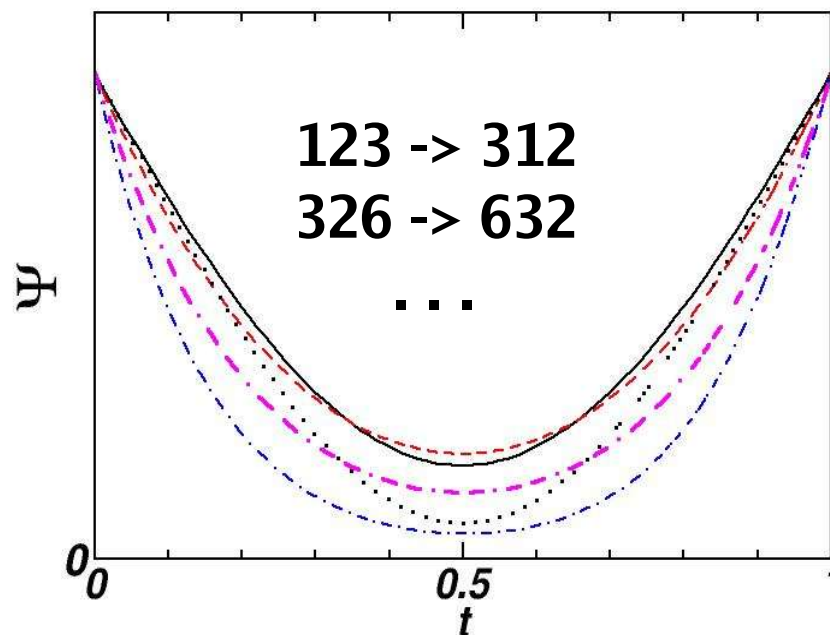
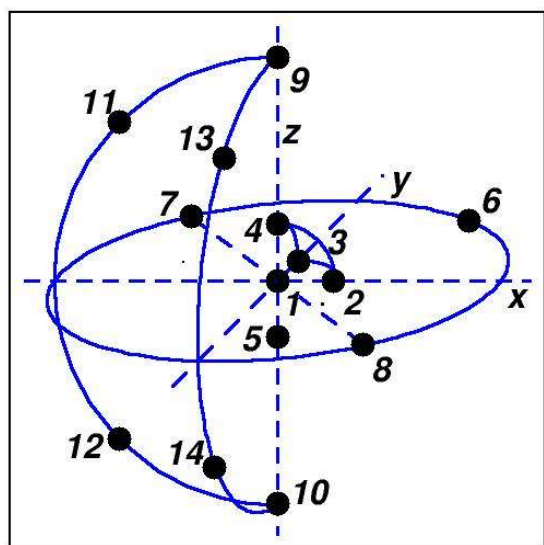
Atomic states (different radial orbitals for subshells):

Proof of two cells for nonint. **and HF** wavefunctions

- position subshells of electrons onto spherical surfaces: explicit factorization

$$\Psi_{HF} = \Psi_{1s} \Psi_{2s2p^3} \Psi_{3s3p^3d^5} \dots$$

- exchanges between the subshells: simple numerical proof up to size $15S(1s2s2p^33s3p^33d^5)$ and beyond (n=4 subshell)



For noninteracting/HF systems adding another spin channel or imposing additional symmetries generate more nodal cells

Unpolarized noninteracting/HF systems: $2*2=4$ nodal cells,
-> product of two independent Slater determinants

$$\psi_{HF} = \det^{\uparrow} \{ \phi_{\alpha} \} \det^{\downarrow} \{ \phi_{\beta} \}$$

- in general, imposing symmetries generates more nodal cells:
the lowest quartet of S symmetry $^4S(1s2s3s)$ has six nodal cells

What happens when interactions are switched on ?

“Nodal/topological degeneracy” is lifted and multiple nodal cells fuse into the minimal two again!

First time showed on Be atom, Bressanini et al '03

Illustrate the general proof idea on a singlet of *interacting* harmonic fermions with BCS as variational wave function

Consider 6 harmonic 2D fermions in the singlet ground state. Rotation by π exchanges two particles in each spin channel, they lie on HF node:

$$\psi_{HF} = 0$$

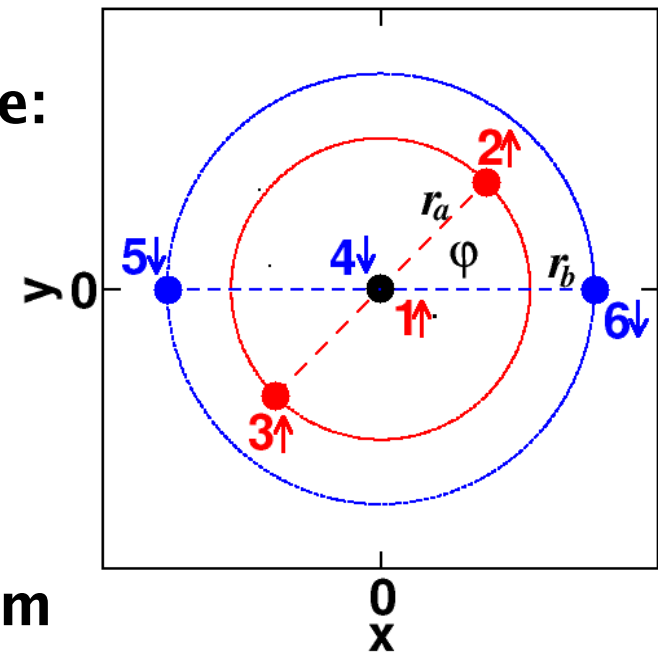
Describe the correlation using the BCS wavefunction:

$$\phi(i, j) = \phi_{HF}(i, j) + \alpha \phi_{corr}(i, j)$$

pairing function includes the virtuals from the first unoccupied subshell

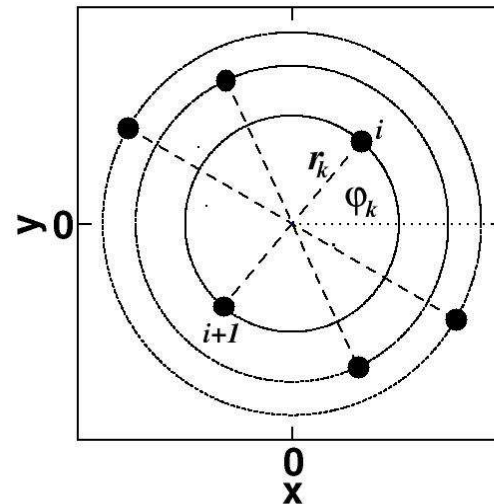
$$\psi_{BCS} = \det\{\phi(i, j)\} = \alpha r_a r_b \cos(\phi) [2(r_a r_b \cos(\phi))^2 - r_a^2 - r_b^2] \neq 0$$

Nonvanishing for arbitrarily weak interaction!



Correlation in the BCS wavefunction is enough to fuse the noninteracting four cells into the minimal two

Arbitrary size: position the particles on HF node (wf. is rotationally invariant)



HF pairing (sum over occupieds, linear dependence in Sl. dets)

$$\psi_{HF} = \det[\phi_{HF}(i, j)] = \det\left[\sum_n \psi_n(i) \psi_n(j)\right] = \det[\psi_n(i)] \det[\psi_n(j)] = 0$$

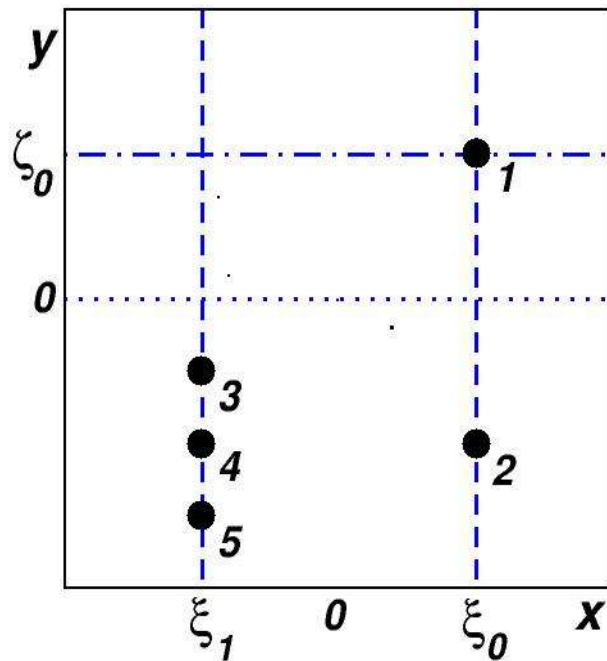
BCS pairing (sum over occupieds and **virtualls, eliminate lin. dep.**)

$$\phi(i, j) = \phi_{HF}(i, j) + \phi_{corr}(i, j)$$

$$\psi_{BCS} = \det[\phi_{BCS}(i, j)] \neq \det[\psi_{nm}(i)] \det[\psi_{nm}(j)] \rightarrow \psi_{BCS} \neq 0$$

Homogeneous electron gas: the spin-up and -down subspaces are interconnected as well; other interacting models

$$r_1^\uparrow = r_6^\downarrow, \quad r_2^\uparrow = r_7^\downarrow, \dots, r_5^\uparrow = r_{10}^\downarrow$$



Translation by $L/2$ in y direction exchanges

$$r_1^\uparrow \longleftrightarrow r_2^\uparrow \quad r_5^\downarrow \longleftrightarrow r_6^\downarrow$$

Translational invariance implies that the wavefunction is constant

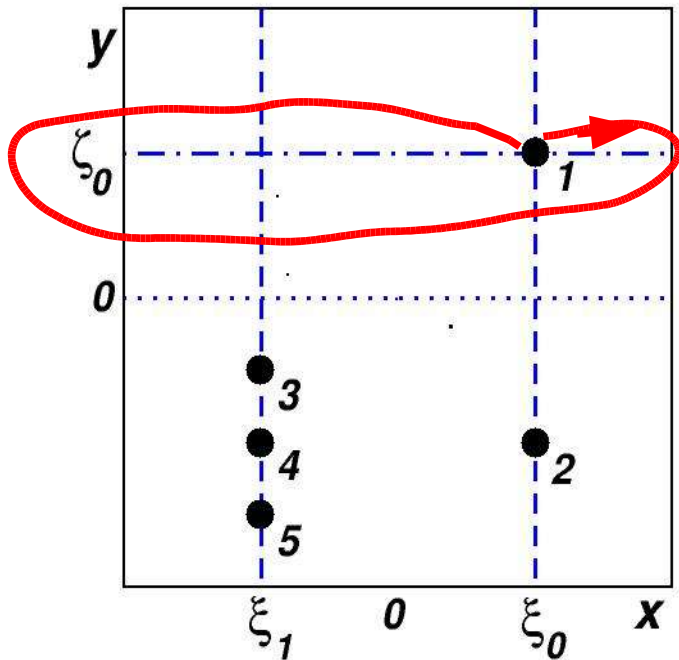
$$\psi_{HF} = 0$$

$$\psi_{BCS} \neq 0$$

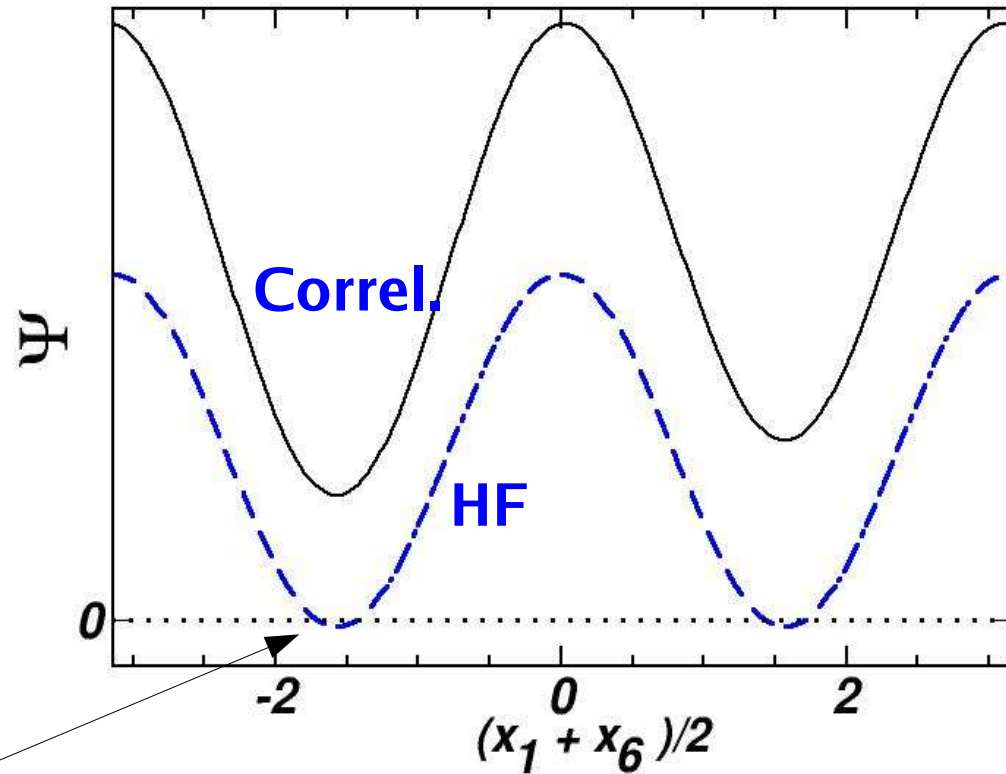
Other interacting models: similar construction

Correlation in homogeneous electron gas: singlet pair of e- winds around the box without crossing the node

$$r_i^\uparrow = r_{i+5}^\downarrow + \text{offset}, \quad i=1, \dots, 5$$



Wavefunction along the winding path



HF crosses the node multiple times, BCS does not (supercond.)

The same applies to the nodes of temperature/imaginary time **density matrix**

Analogous argument applies to temperature density matrix

$$\rho(R, R', \beta) = \sum_{\alpha} \exp[-\beta E_{\alpha}] \psi_{\alpha}^{*}(R) \psi_{\alpha}(R')$$

fix R', β -> nodes/cells in the R subspace

At high (classical) temperatures

$$\rho(R, R', \beta) = C_N \det \left\{ \exp \left[-\frac{(r_i - r'_j)^2}{2\beta} \right] \right\}$$

It is not too difficult to prove that at classical temperatures R and R' subspaces have only two nodal cells: it is stunning since there is a summation over the whole spectrum!

PRL, 96, 240402 /cond-mat/0601485 (the basic ideas)
cond-mat/0605550 (all the models, density matrix)

Two nodal cells: generic property, possible counterexamples

Also, how about the exact shape of the node ?

Topology of the nodes closed-shell ground states is surprisingly simple:

The ground state node bisects the configuration space (the most economic way to satisfy the antisymmetry)

Possible exceptions:

- nonlocal interactions, strong interactions
- impose more symmetries or boundaries
- large degeneracies

But the exact shape very difficult to get

- mostly through wavefunction improvement methods

Beyond single particle orbitals: singlet pair orbital Bardeen-Cooper-Schrieffer (BCS) wavefunction

- used to describe superconductivity or BEC, Sorella et al for electronic structure, '04
antisymmetrized product of singlet pair orbitals $\phi(i, j)$

$$\psi_{BCS} = A[\phi(i, j)] = \det[\phi(i, j)]$$

- **spin-polarized case:** $N^\downarrow = n$ while $N^\uparrow = n + o$

$$\psi_{BCS} = A[\phi(1, n) \dots \phi(n, 2n) \times h_1(2n+1) \dots h_o(2n+o)]$$

where $h_k(i)$ are one-particle orbitals (usually HF)

- fully spin-polarized state trivially recovers Hartree-Fock

$$\psi_{BCS} = A[h_i(j)] = \det[h_i(j)] = \psi_{HF}$$

Beyond Slater determinants: pfaffian pairing wavefunctions contain **both singlet and triplet pairs -> all spin states treated consistently**

$$\psi_{PF} = pf \begin{bmatrix} \chi^{\uparrow\uparrow} & \phi^{\uparrow\downarrow} & \psi^{\uparrow} \\ -\phi^{\uparrow\downarrow T} & \chi^{\downarrow\downarrow} & \psi^{\downarrow} \\ -\psi^{\uparrow T} & -\psi^{\downarrow T} & 0 \end{bmatrix} \times \exp[U_{corr}]$$

- pairing orbitals expanded in one-particle basis

$$\phi(i, j) = \sum_{\alpha \geq \beta} a_{\alpha\beta} [h_{\alpha}(i)h_{\beta}(j) + h_{\beta}(i)h_{\alpha}(j)]$$

$$\chi(i, j) = \sum_{\alpha > \beta} b_{\alpha\beta} [h_{\alpha}(i)h_{\beta}(j) - h_{\beta}(i)h_{\alpha}(j)]$$

- unpaired $\psi(i) = \sum_{\alpha} c_{\alpha} h_{\alpha}(i)$

- expansion coefficients and the Jastrow correlation optimized with respect to energy

Algebra of Determinants vs Pfaffians

Determinant

- signed sum of all antisymmetric permutations of rows/collums
- square matrix $N \times N$
- for $N=2$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- expansion by minors
- for any square matrix B

$$\det(B) = (-1)^{n(n-1)/2} \text{pf} \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}$$

Pfaffian

- signed sum of all antisymmetric perm. of pairs of elements
- skew-symmetric matrix ($N=2n$)

$$\text{pf} \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} =$$

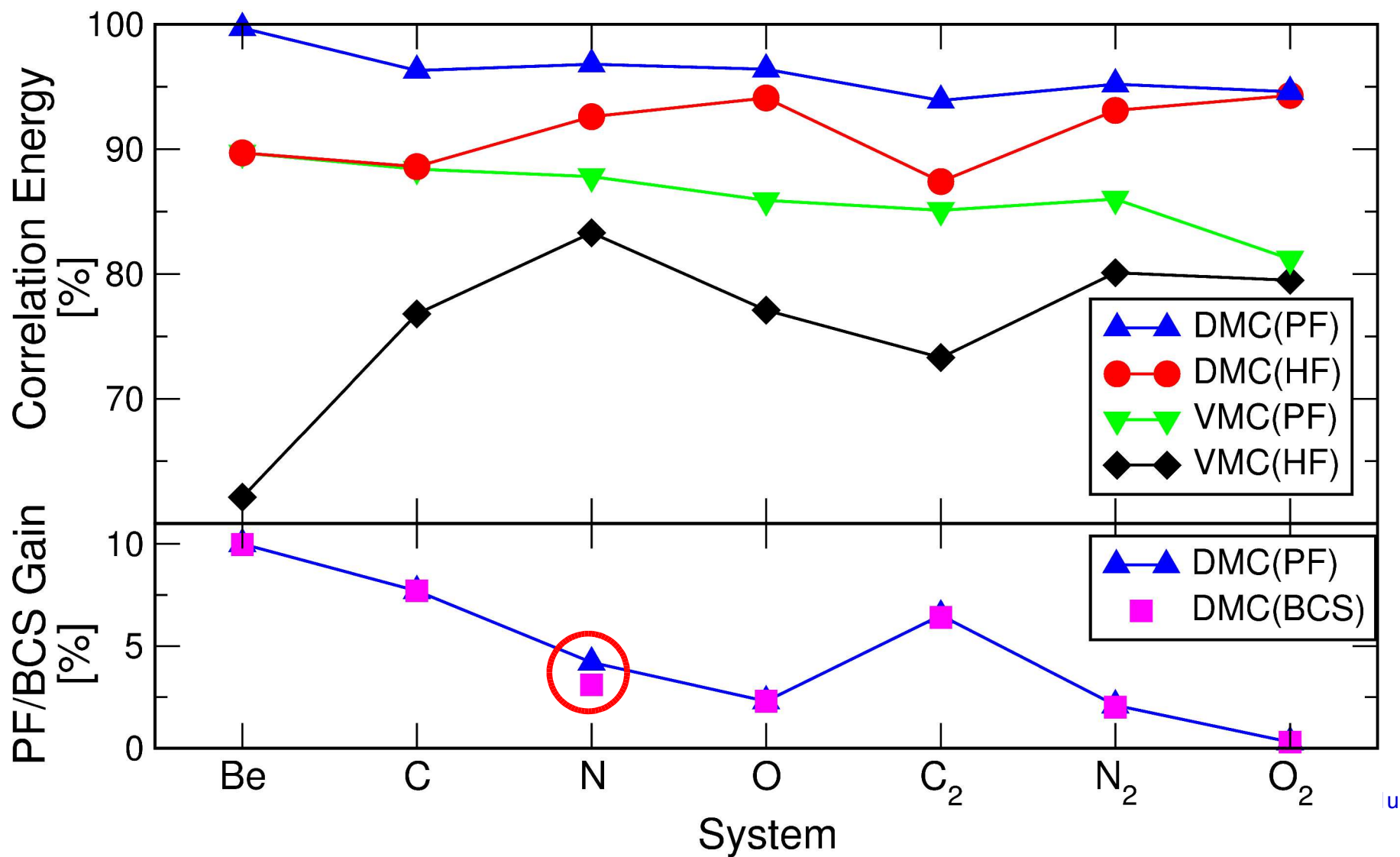
$$= [a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}]$$

- expansion by pfaffian minors
- for any skew-symmetric matrix A

$$\det(A) = [\text{pf}(A)]^2$$

Correlation energies of first row atoms and dimers

Correlation from singlet vs triplet pairing



Multi-pfaffian wavefunctions for first row atoms: FNDMC ~98-99 % of correlation with a few pfaffians!

Table of % of correlation energies recovered for CI vs MPF w.f.
 - n denotes the number of dets/pfs in the expansion

WF	n	C	n	N	n	O
VMC(MPF)	3	92.3(1)	5	90.6(1)	11	92.6(3)
VMC(CI)	98	89.7(4)	85	91.9(2)	136	89.7(4)
DMC(MPF)	3	98.9(2)	5	98.4(1)	11	97.2(1)
DMC(CI)	98	99.3(3)	85	98.9(2)	136	98.4(2)

- number of pfaffians n
 - subject to symmetry constraints
 - in principle all distinct pairs could be included

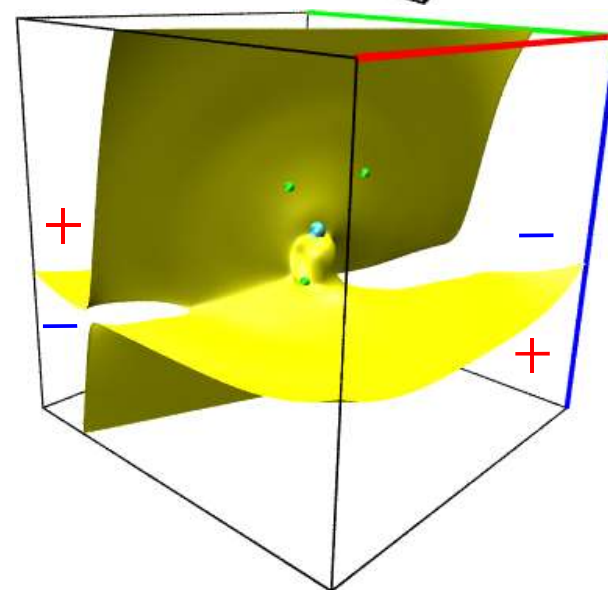
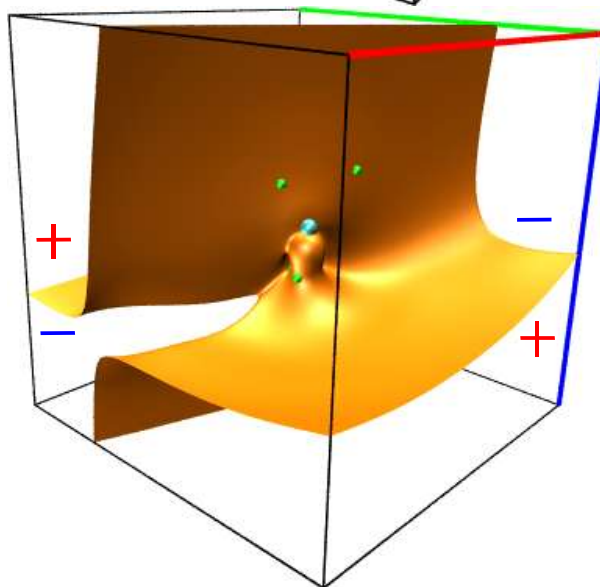
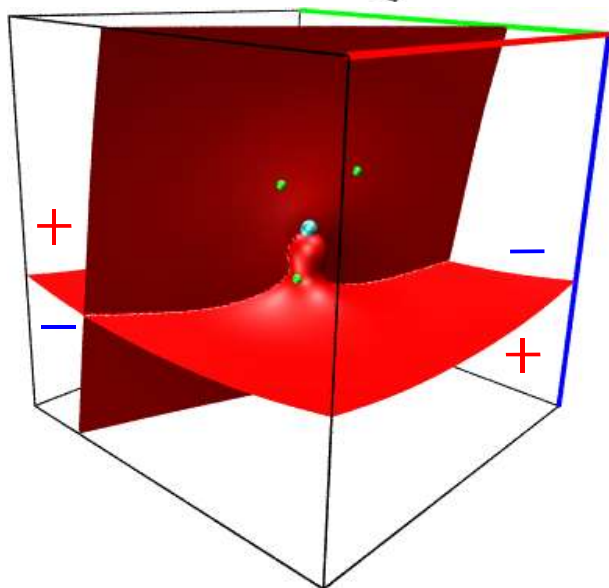
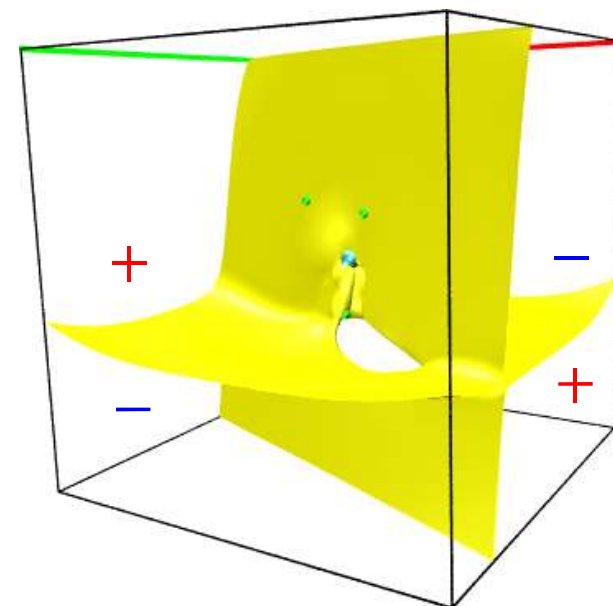
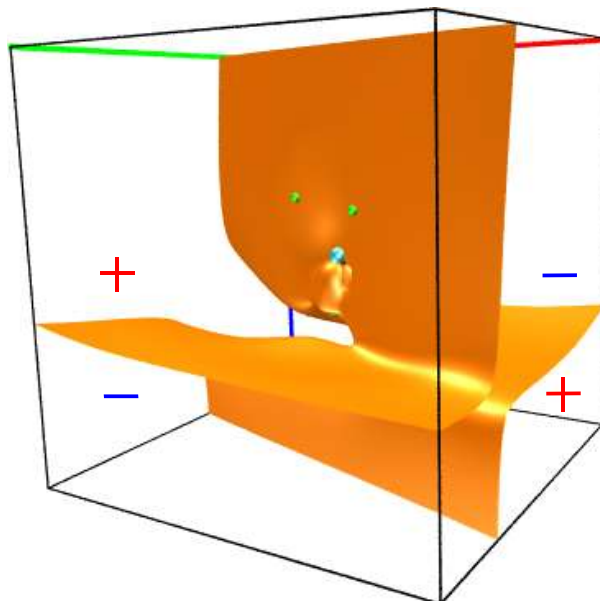
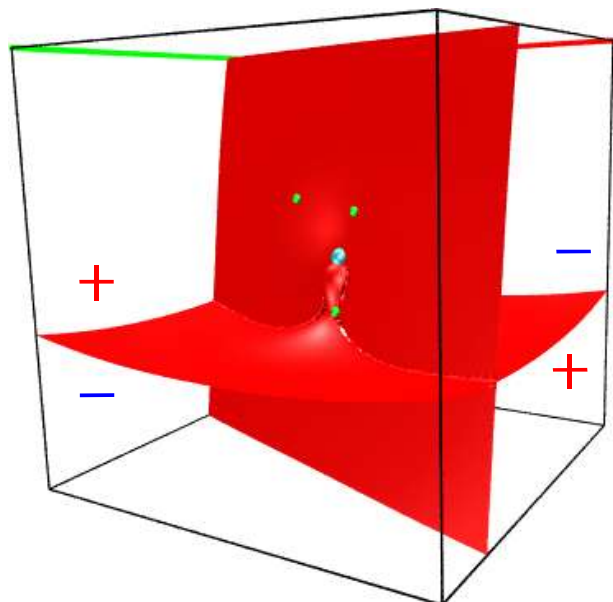
M. Bajdich et al, PRL 96, 130201 (2006)

3D scan of the oxygen atom node by 2e- singlet: Topologies of different wfs (fixed-node DMC Ecorr)

HF (94.0(2)%)

MPF (97.4(1)%)

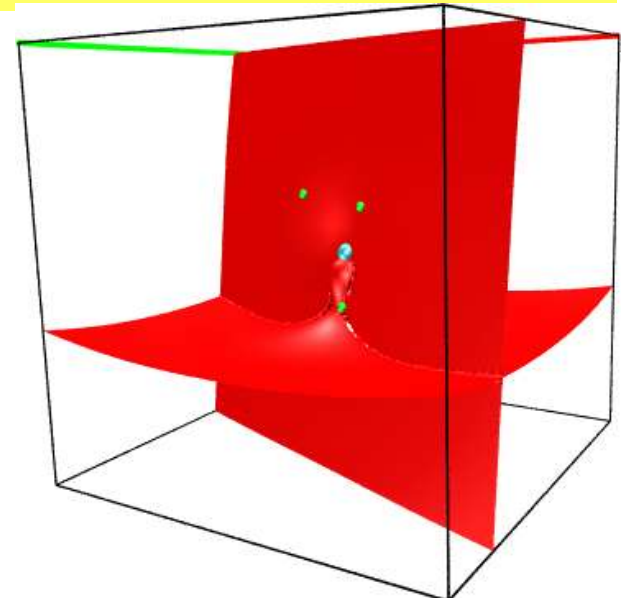
CI (99.8(3)%)



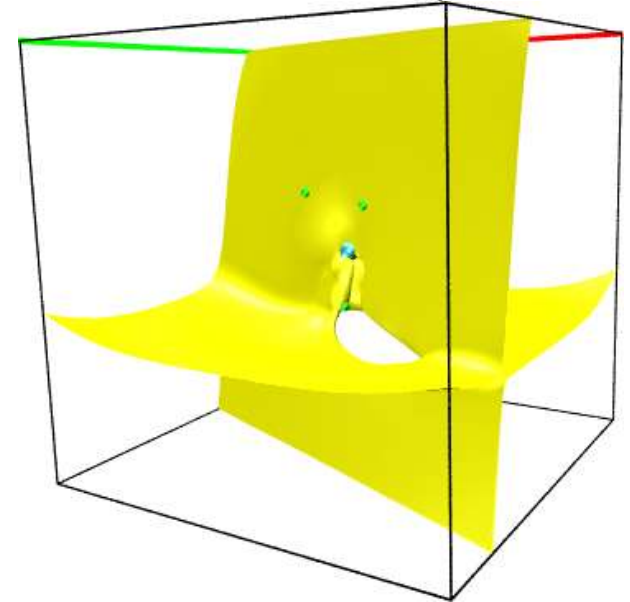
Observations from comparison of HF and “exact” nodes

- the two nodal cells for Coulomb interactions as well
- the nodal openings have rather fine structure
- openings are important -> ~ 5% of the correlation energy
- although topologically incorrect, away from openings HF nodes unexpectedly close to exact

HF



CI



Summary

- **explicit proof** that, in general, fermionic ground states and density matrices have **two nodal cells for $d > 1$** and for any size - **fundamental property of fermionic systems**
- **nodal openings** in correlated wave functions and **exact nodal shape important: 5 % of correlation energy**, necessary condition for superconductivity; pfaffians pairing wfs very efficient
- **counterexamples**: multiple cells can be genuine, eg, from singular or nonlocal interactions, boundary conditions, possibly by large degeneracies, etc
- fermion nodes: another example of importance of **quantum geometry (field theory) and topology** for electronic structure

Fermion nodes summary

- fermion nodes not that mysterious; for most ground states just bisection of the many-body space into two parts
- fermion node: another manifestation of importance of **quantum geometry and topology** (special case of a phase)
- nodes directly related to **important many-body effects**; e.g., phase change of the electronic system (superfluidity)
- pair orbitals improve the wavefunctions but the fixed-node error is still significant

New developments: coupling of QMC with ab initio molecular dynamics -> QMC/MD !

- so far QMC used only for static, state-by-state calculations
- Car-Parrinello MD:
 - ions evolve according to classic EOM
 - electrons with Density Functional Theory

Typical displacement of an ion in one MD step

10^{-3} to 10^{-4} a.u.

Typical displacement of an e- in one DMC step

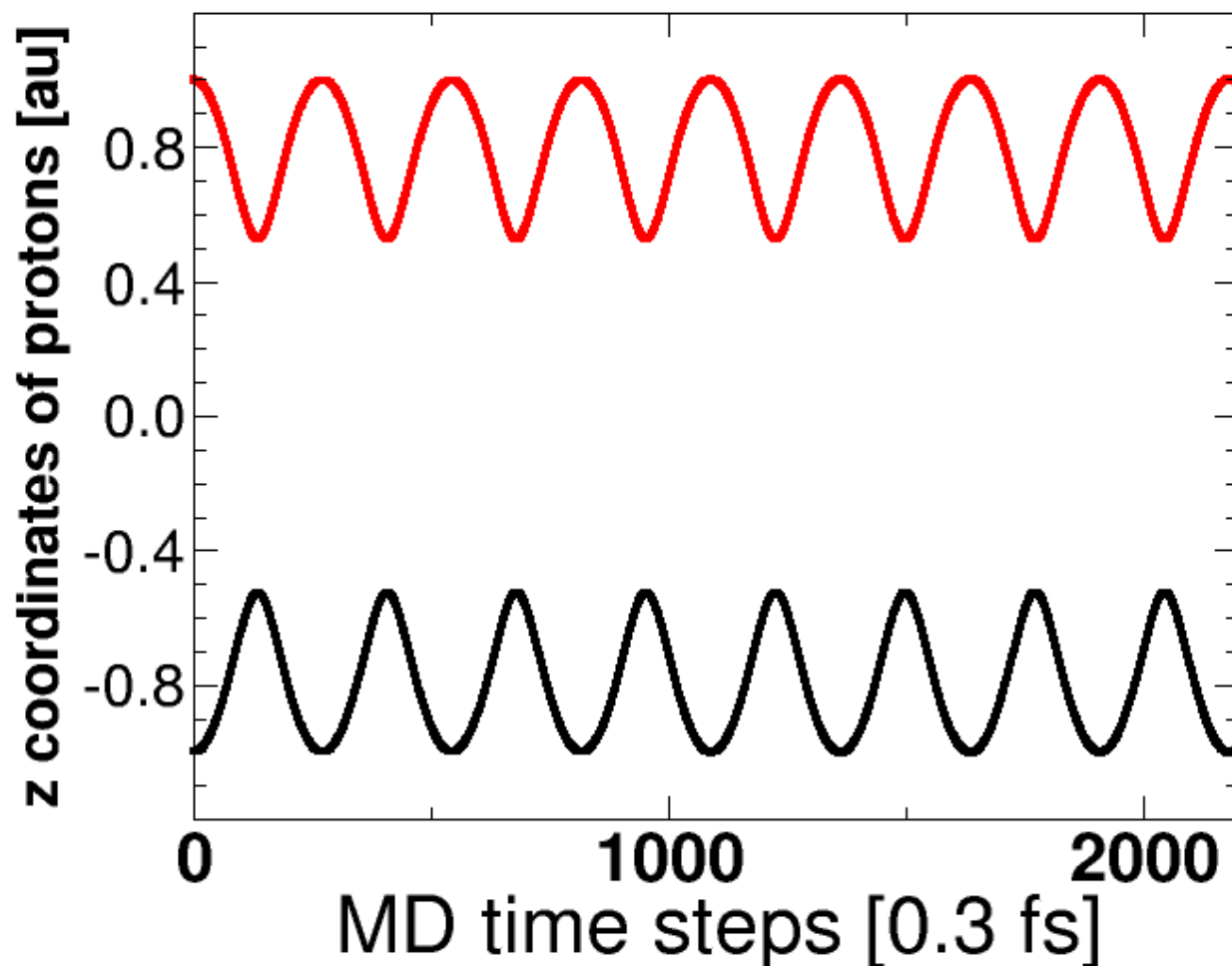
10^{-1} to 10^{-2} a.u.

Key idea: QMC walkers are fast, couple the evolution of ions with the evolution of the wavefunction (factor 50 in efficiency!)

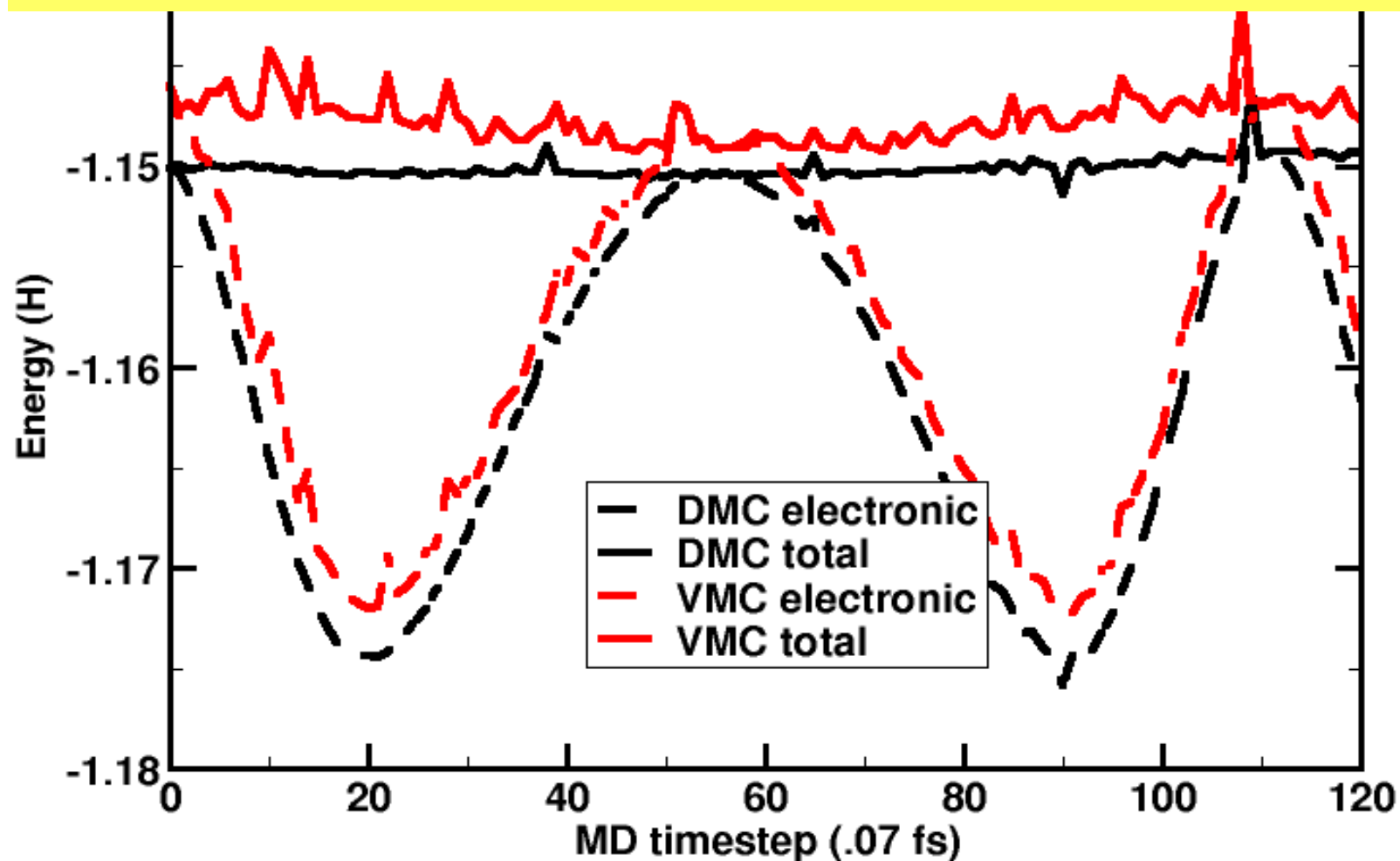
First calculations with QMC/MD method

H2 vibration using QMC forces

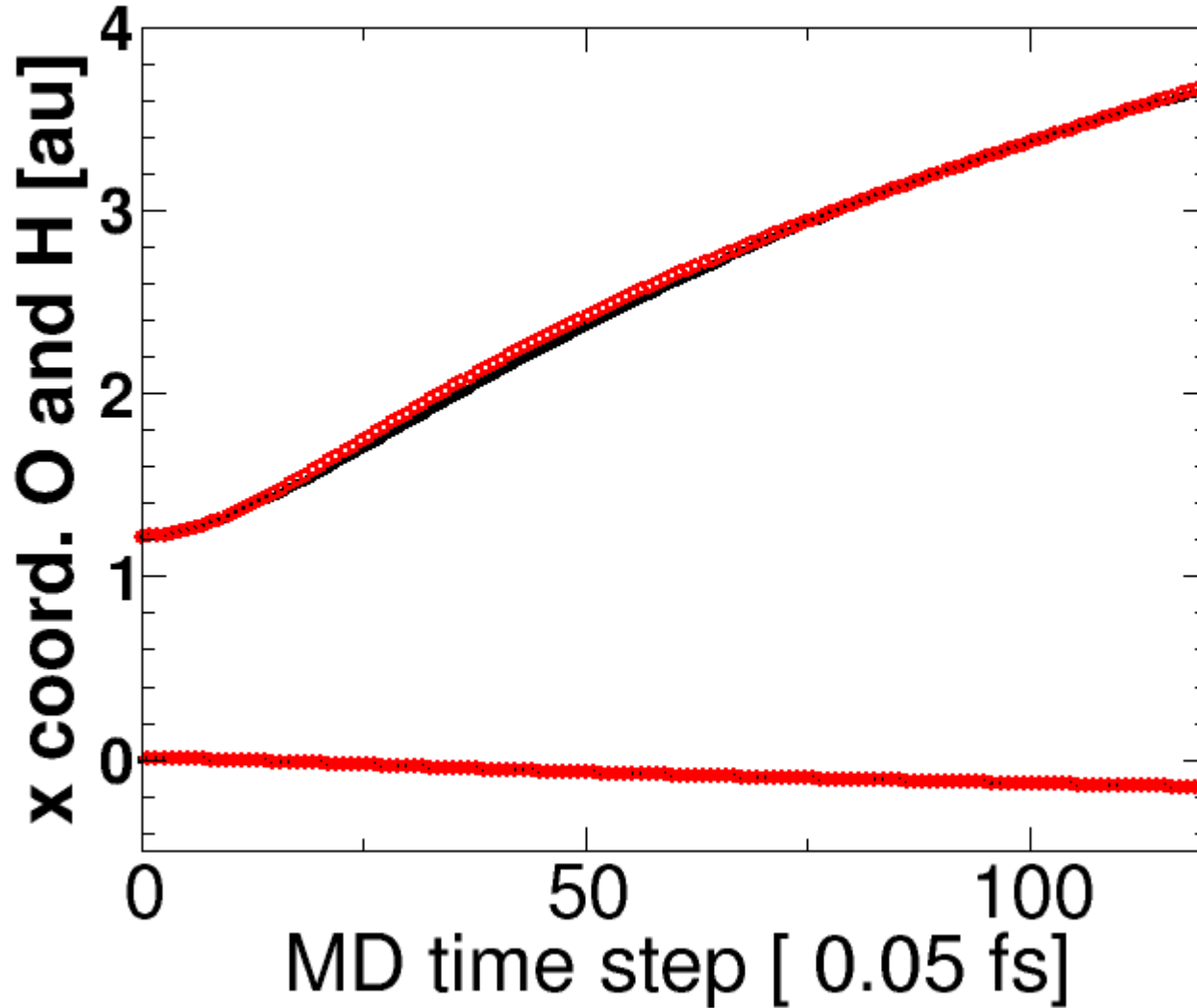
t_MD=0.3 fs, opt 610, f 10/40/4, dec 6/10/4



H2 vibration: total and electronic energies in VMC and DMC



Dynamical dissociation of H₂O molecule: QMC/MD vs Car-Parrinello AIMD



QMC as an alternative for electronic structure

- focus on efficient description many-body effects, put the many-body effects where they belongs: to the wavefunction (new wave function was always a milestone: Hartree-Fock, BCS, Laughlin...)
- one still has to do the physics: which types of correlations, symmetries, phases, . . ., the fundamental and fun part!
- but: tedious integrals, averaging, etc, left to machines
- gives a good use to machines: scales $N^{(1-3)}$ with the number of e- and efficient on large parallel platforms (the same code on my desktop and on LLNL thousand processors), robust
- often the most accurate method available: benchmarks
- opens new perspectives on many-body quantum phenomena

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